A CONVERGENT POINT INTEGRAL METHOD FOR ISOTROPIC ELLIPTIC EQUATIONS ON A POINT CLOUD*

ZHEN LI† AND ZUOQIANG SHI†

Abstract. In this paper, we propose a numerical method to solve isotropic elliptic equations on a point cloud by generalizing the point integral method. The idea of the point integral method is to approximate the differential operators by integral operators and discretize the corresponding integral equation on a point cloud. The key step is to get the integral approximation. In this paper, with proper kernel function, we get an integral approximation for the elliptic operators with isotropic coefficients. Moreover, the integral approximation has been proved to keep the coercivity of the original elliptic operator. The convergence of the point integral method is also proved.

1. Introduction. Today, data plays more and more important roles in science and engineering. In many problems, data is usually represented as a collection of points embedding in a high dimensional Euclidean space. Processing and analysis of the point cloud data is essential in many applications, such as machine learning [4, 11] and image processing [31, 30].

In many applications, the point cloud data lies in a manifold whose dimension is much lower than the ambient Euclidean space. The low dimensionality is an important feature we could exploit to analyze the data. One example is the low dimensional manifold model (LDMM) in image processing [30]. In this model, the original image is cut into many overlapping patches. The collection of all patches consists of a point cloud in Euclidean space. It is found that for many natural images, the patch set usually samples a low dimensional manifold which is called the patch manifold. The dimension of the patch manifold is used as a regularization to processing the image. Based on differential geometry and a variational method, this model is reduced to solve the Laplace equation on patch set. The key point in LDMM is to solve this Laplace equation accurately and efficiently.

Beside the data analysis, solving PDEs on a manifold also appears in many physical problems, such as material science [9, 17], fluid flow [19, 21], and biology and biophysics [3, 18, 29, 2]. To solve PDEs on a manifold, many methods have been developed, especially on two dimensional (2D) surfaces, including the surface finite element method [16], level set method [8, 37], grid based particle method [25, 24], and closest point method [32, 28]. However, these methods need extra information besides the point cloud, for instance, meshes, level set function, and closest point function. This information is not easy to obtain from a point cloud when the dimension of the manifold is high.

Recently, Liang and Zhao proposed to discretize the differential operators on a point cloud by local least square approximations of the manifold [27]. Their method

---

†Yau Mathematical Sciences Center, Tsinghua University, Beijing, China, 100084 (zli12@mails.tsinghua.edu.cn, zqshi@math.tsinghua.edu.cn).

can achieve a high order of accuracy and enjoy more flexibility since no mesh is needed. In principle, it can be applied to manifolds with arbitrary dimensions and co-dimensions with or without boundary. However, if the dimension of the manifold is high, this method may not be stable since a high order polynomial is used to fit the data. Later, Lai, Liang, and Zhao proposed a local mesh method to approximate the differential operators on a point cloud [23]. The main idea is to construct the mesh locally around each point by using K nearest neighbors. The local mesh is easier to construct than the global mesh. Based on the local mesh, it is easy to discretize differential operators and compute integrals. However, when the dimension of the manifold is high, even the local mesh is not easy to construct.

The original point integral method (PIM) for Laplace equation is closely related with the graph Laplacian [10, 7]. Graph Laplacian has been widely used in many problems. It is observed in [5, 22, 20, 35] that the graph Laplacian with the Gaussian weights well approximates the Laplace–Beltrami operator when the vertices of the graph are assumed to sample the underlying manifold. When there is no boundary, Belkin and Niyogi [6] showed the spectra of the graph Laplacian with Gaussian weights converges to that of Laplace–Beltrami operator. Recently, Singer and Wu [36] showed the spectral convergence of the graph Laplacian in the presence of the Neumann boundary.

Inspired by the graph Laplacian and the nonlocal diffusion, we developed the PIM for the Poisson equation on a point cloud [26, 33, 34].

$$-\Delta_{\mathcal{M}}u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \mathcal{M},$$

where $\Delta_{\mathcal{M}} = \mathrm{div}(\nabla)$ is the Laplace–Beltrami operator in $\mathcal{M}$.

We assume that $\mathcal{M} \in C^{\infty}$ is a compact k-dimensional manifold isometrically embedded in $\mathbb{R}^d$ with the standard Euclidean metric and $k \leq d$. If $\mathcal{M}$ has boundary, the boundary $\partial\mathcal{M}$ is also a C^{∞} smooth manifold.

Let $\Phi : \Omega \subset \mathbb{R}^k \to \mathcal{M} \subset \mathbb{R}^d$ be a local parametrization of $\mathcal{M}$ and $\theta \in \Omega$. For any differentiable function $f : \mathcal{M} \to \mathbb{R}$, define the gradient on the manifold

$$(1.1) \qquad \nabla f(\Phi(\theta)) = \sum_{i,j=1}^{m} g^{ij}(\theta)\frac{\partial\Phi}{\partial\theta_i}(\theta)\frac{\partial f(\Phi(\theta))}{\partial\theta_j}(\theta),$$

and for vector field $F : \mathcal{M} \to T_{\mathbf{x}}\mathcal{M}$ on $\mathcal{M}$, where $T_{\mathbf{x}}\mathcal{M}$ is the tangent space of $\mathcal{M}$ at $\mathbf{x} \in \mathcal{M}$, the divergence is defined as

$$(1.2) \qquad \mathrm{div}(F) = \frac{1}{\sqrt{\det G}}\sum_{k=1}^{d}\sum_{i,j=1}^{m}\frac{\partial}{\partial\theta_i}\left(\sqrt{\det G}g^{ij}F^k(\Phi(\theta))\frac{\partial\Phi^k}{\partial\theta_j}\right),$$

where $(g^{ij})_{i,j=1,\ldots,k} = G^{-1}$, $\det G$ is the determinant of matrix G, and $G(\theta) = (g_{ij})_{i,j=1,\ldots,k}$ is the first fundamental form which is defined by

$$(1.3) \qquad g_{ij}(\theta) = \sum_{k=1}^{d}\frac{\partial\Phi_k}{\partial\theta_i}(\theta)\frac{\partial\Phi_k}{\partial\theta_j}(\theta), \quad i,j = 1,\ldots,m,$$

and $(F^1(\mathbf{x}),\ldots,F^d(\mathbf{x}))^t$ is the representation of F in the embedding coordinates.

The main idea of the PIM is to approximate the Poisson equation by the following integral equation:

$$-\int_{\mathcal{M}}\Delta_{\mathcal{M}}u(\mathbf{y})\bar{R}_t(\mathbf{x},\mathbf{y})d\mu_{\mathbf{y}} \approx \frac{1}{t}\int_{\mathcal{M}}R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})$$
$$- u(\mathbf{y}))d\mu_{\mathbf{y}} - 2\int_{\partial\mathcal{M}}\bar{R}_t(\mathbf{x},\mathbf{y})\frac{\partial u}{\partial\mathbf{n}}(\mathbf{y})d\tau_{\mathbf{y}},$$

where $\mathbf{n}$ is the out normal of $\mathcal{M}$, and $R_t(\mathbf{x}, \mathbf{y})$ and $\bar{R}_t(\mathbf{x}, \mathbf{y})$ are kernel functions given as follows:

$$(1.4) \qquad R_t(\mathbf{x}, \mathbf{y}) = C_t R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t}\right), \quad \bar{R}_t(\mathbf{x}, \mathbf{y}) = C_t \bar{R}\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t}\right),$$

where $C_t = \frac{1}{(4\pi t)^{k/2}}$ is the normalizing factor, $R \in C^2(\mathbb{R}^+)$ is a positive function which is integrable over $[0, +\infty)$, and

$$\bar{R}(r) = \int_r^{+\infty} R(s)\mathrm{d}s.$$

There are not any derivatives in the integral equation. It is easy to be discretized from point clouds using some quadrature rule. In [33, 34], we proved the convergence of the PIM for the Poisson equation with the Neumann and Dirichlet boundary condition.

In the PIM, we only need the point cloud to discretize the differential operator. This gives PIM great flexibility to fit the requirements in a variety of applications. However, one limitation of the PIM is that it only applies on a Laplace–Beltrami operator. In many problems, we need to discretize other differential operators besides a Laplace–Beltrami operator. In this paper, we generalize the PIM to isotropic elliptic operators. Isotropic elliptic operators are also widely used in many problems. One example is the nonlocal total variation minimization on a point cloud, in which we need to solve an optimization problem,

$$\min_u \|\nabla u\|_{L^1(\mathcal{M})}, \quad \text{subject to} \quad \Psi(u) = b,$$

where ∇ is the gradient in $\mathcal{M}$, Ψ is the measurement operator related with the application, b is the observation, and

$$\|\nabla u\|_{L^1(\mathcal{M})} = \int_{\mathcal{M}} |\nabla u(\mathbf{x})|\mathrm{d}\mathbf{x}.$$

Using a standard variational approach, the solution of above optimization problem can be given by solving a nonlinear elliptic equation,

$$-\mathrm{div}\left(\frac{\nabla u(\mathbf{x})}{|\nabla u(\mathbf{x})|}\right) = f(\mathbf{x}),$$

where $f(\mathbf{x})$ is a known function. Apparently, this equation can be solved by solving a sequence of isotropic elliptic equation iteratively.

In this paper, we consider to solve elliptic equations with isotropic coefficients on manifold $\mathcal{M}$,

$$(1.5) \qquad -\mathrm{div}(p^2(\mathbf{x})\nabla u(\mathbf{x})) = f(\mathbf{x}), \quad \mathbf{x} \in \mathcal{M}.$$

The coeffcients $p(\mathbf{x})$ and source term $f(\mathbf{x})$ are known smooth functions of spatial variables, i.e.,

$$p \in C^1(\mathcal{M}), \ f \in C^1(\mathcal{M}).$$

The elliptic condition makes it so that there exist generic constants $c_0, c_1 > 0$ such that for any $\mathbf{x} \in \mathcal{M}$,

$$0 < c_0 \le p(\mathbf{x}) \le c_1 < \infty.$$

The key observation in this paper is the integral approximation of isotropic elliptic operators given as

$$(1.6) \quad -\int_{\mathcal{M}} \text{div}(p^2(\mathbf{y})\nabla u(\mathbf{y})) \frac{\bar{R}_t(\mathbf{x},\mathbf{y})}{p(\mathbf{y})} d\mu_{\mathbf{y}} \approx \frac{1}{t} \int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x}) - u(\mathbf{y}))p(\mathbf{y}) d\mu_{\mathbf{y}}$$
$$- 2\int_{\partial\mathcal{M}} \frac{\partial u}{\partial \mathbf{n}}(\mathbf{y}) \bar{R}_t(\mathbf{x},\mathbf{y})p(\mathbf{y}) d\tau_{\mathbf{y}},$$

where the kernel functions R_t and $\bar{R}_t$ are the same as those in (1.4). The main advantage of this integral approximation is that there is no differential operator inside. Using this approximation, we transfer the numerical differential to a numerical integral which is much easier to compute on a point cloud. Based on this integral approximation, we are able to develop the PIM to isotropic elliptic equations.

Similar integral approximation is also widely used in nonlocal diffusion and the peridynamic model [12, 1, 13, 14, 38]. The integral approximation is easy to implement on a point cloud, since it has no derivatives inside. Moreover, the PIM also has a very good theoretical property. It is proved that the coercivity of the original elliptic operator is partially preserved, and this partial coercivity implies the convergence of the PIM.

The rest of the paper is organized as follows. In section 2, we introduce the PIM for isotropic elliptic operator with the Neumann and Dirichlet boundary condition. The convergence analysis is given in section 3. Several numerical examples are presented in section 4. The concluding remarks are made in section 5.

2. Point integral method for isotropic elliptic equations. In this section, we introduce a numerical method for isotropic elliptic equations on a point cloud based on the integral approximation (1.6).

To simplify the notation, we introduce an integral operator,

$$(2.1) \qquad L_t u(\mathbf{x}) = \frac{1}{t} \int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x}) - u(\mathbf{y}))p(\mathbf{y}) d\mu_{\mathbf{y}},$$

where R_t is the kernel function given in (1.4).

2.1. Neumann boundary. First, we consider the Neumann problem,

$$(2.2) \qquad \begin{cases} -\text{div}(p^2(\mathbf{x})\nabla u(\mathbf{x})) = f(\mathbf{x}), & \mathbf{x} \in \mathcal{M}, \\ \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = g(\mathbf{x}), & \mathbf{x} \in \partial\mathcal{M}. \end{cases}$$

Using the integral approximation (1.6), the solution of the Neumann problem (2.2) can be obtained approximately by solving an integral equation

$$(2.3) \qquad L_t u(\mathbf{x}) - 2\int_{\partial\mathcal{M}} g(\mathbf{y}) \bar{R}_t(\mathbf{x},\mathbf{y})p(\mathbf{y}) d\tau_{\mathbf{y}} = \int_{\mathcal{M}} f(\mathbf{y}) \frac{\bar{R}_t(\mathbf{x},\mathbf{y})}{p(\mathbf{y})} d\mu_{\mathbf{y}}$$

with $t \ll 1$.

The eigenvalue problem is also solved by a generalized eigenvalue problem

$$(2.4) \qquad L_t u(\mathbf{x}) = \lambda \int_{\mathcal{M}} u(\mathbf{y}) \frac{\bar{R}_t(\mathbf{x},\mathbf{y})}{p(\mathbf{y})} d\mu_{\mathbf{y}}.$$

2.2. Dirichlet boundary. The Dirichlet problem is more involved in the PIM, since the normal derivative $\frac{\partial u}{\partial \mathbf{n}}$ is not known.

$$(2.5) \qquad \begin{cases} -\text{div}(p^2(\mathbf{x})\nabla u(\mathbf{x})) = f(\mathbf{x}), & \mathbf{x} \in \mathcal{M}, \\ u(\mathbf{x}) = g(\mathbf{x}), & \mathbf{x} \in \partial\mathcal{M}. \end{cases}$$

Here, we use the same idea as that in [26] to deal with the Dirichlet boundary.

2.2.1. Robin approximation. The simplest way is using Robin boundary to approximate the Dirichlet boundary. More specifically, we consider the following Robin problem:

$$(2.6) \qquad \begin{cases} -\text{div}(p^2(\mathbf{x})\nabla u(\mathbf{x})) = f(\mathbf{x}), & \mathbf{x} \in \mathcal{M}, \\ u(\mathbf{x}) + \beta \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = g(\mathbf{x}), & \mathbf{x} \in \partial\mathcal{M}, \end{cases}$$

where $0 < \beta \ll 1$ is a small parameter. It is easy to show that as $\beta \to 0$, the solution of the Robin problem, (2.6), converges to the solution of the Dirichlet problem, (2.5).

For the Robin problem, the integral approximation (1.6) is applicable to give an integral equation,

$$(2.7) \qquad L_t u(\mathbf{x}) - \frac{2}{\beta} \int_{\partial\mathcal{M}} (g(\mathbf{y}) - u(\mathbf{y}))\bar{R}_t(\mathbf{x}, \mathbf{y})p(\mathbf{y})\mathrm{d}\tau_{\mathbf{y}} = \int_{\mathcal{M}} \frac{f(\mathbf{y})}{p(\mathbf{y})} \bar{R}_t(\mathbf{x}, \mathbf{y})\mathrm{d}\mu_{\mathbf{y}}.$$

Similarly, we also get an approximation of the eigenvalue problem,

$$(2.8) \qquad L_t u(\mathbf{x}) + \frac{2}{\beta} \int_{\partial\mathcal{M}} u(\mathbf{y})\bar{R}_t(\mathbf{x}, \mathbf{y})p(\mathbf{y})\mathrm{d}\tau_{\mathbf{y}} = \lambda \int_{\mathcal{M}} u(\mathbf{y})\frac{\bar{R}_t(\mathbf{x}, \mathbf{y})}{p(\mathbf{y})}\mathrm{d}\mu_{\mathbf{y}}.$$

2.2.2. Iterative solver based on augmented Lagrangian multiplier. In the Robin approximation, the parameter β has to be small to get good approximation, while the linear system becomes ill-conditioned. To alleviate this difficulty, we could use an iterative method based on the augmented Lagrange method (ALM).

It is well known that the Dirichlet problem can be reformulated to be following constrained optimization problem:

$$(2.9) \qquad \min_{v \in H^1(\mathcal{M})} \frac{1}{2} \int_{\mathcal{M}} p^2(\mathbf{y})|\nabla v(\mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{y}} + \int_{\mathcal{M}} f(\mathbf{y})v(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}},$$
$$\text{subject to} \quad v(\mathbf{x})|_{\partial\mathcal{M}} = g(\mathbf{x}).$$

Applying the ALM method to the problem (2.9), we get an iterative method, and in each step, an unconstrained optimization problem is solved,

$$\min_{v} \quad \frac{1}{2} \int_{\mathcal{M}} p^2(\mathbf{y})|\nabla v(\mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{y}} + \int_{\mathcal{M}} f(\mathbf{y}) \cdot v(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}$$
$$(2.10) \qquad + \int_{\partial\mathcal{M}} w^k(\mathbf{y}) \cdot (g(\mathbf{y}) - v(\mathbf{y}))p^2(\mathbf{y})\mathrm{d}\tau_{\mathbf{y}} + \frac{1}{2\beta} \int_{\partial\mathcal{M}} (g(\mathbf{y}) - v(\mathbf{y}))^2 p^2(\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}.$$

Using the variational method, one can show that the solution to (2.10) is exactly the solution to the following Robin problem:

$$(2.11) \qquad \begin{cases} \text{div}(p^2(\mathbf{x})\nabla v(\mathbf{x})) = f(\mathbf{x}), & \mathbf{x} \in \mathcal{M}, \\ v(\mathbf{x}) + \beta \frac{\partial v}{\partial \mathbf{n}}(\mathbf{x}) = g(\mathbf{x}) + \beta w^k(\mathbf{x}), & \mathbf{x} \in \partial\mathcal{M}. \end{cases}$$

This Robin problem is solved by the integral equation. Notice that the parameter β is not necessarily small. Usually, we set $\beta = 1$. Thus, the linear system is not ill-conditioned.

2.3. Discretization. The main advantage of the integral equations is that they are easy to discretize over the point cloud since there are not any derivatives inside.

Assume we are given a set of sample points P sampling the submanifold $\mathcal{M}$ and a subset $S \subset P$ sampling the boundary of $\mathcal{M}$. List the points in P, respectively, S, in a

---

Procedure 1 ALM for Dirichlet Problem

1: $k = 0$, $w^0 = 0$.

2: **repeat**

3:     Solving the following integral equation to get v^k,

$$L_t v^k(\mathbf{y}) - \frac{2}{\beta} \int_{\partial \mathcal{M}} (g(\mathbf{y}) - v^k(\mathbf{y}) + \beta w^k(\mathbf{y})) \bar{R}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}}$$

$$= \int_{\mathcal{M}} \frac{f(\mathbf{y})}{p(\mathbf{y})} \bar{R}_t(\mathbf{x}, \mathbf{y}) \mathrm{d}\mu_{\mathbf{y}}.$$

4:     $w^{k+1} = w^k + \frac{1}{\beta}(g - (v^k|_{\partial \mathcal{M}}))$, $k = k + 1$

5: **until** $\|g - (v^{k-1}|_{\partial \mathcal{M}})\| = 0$

6: $u = v^k$

---

fixed order $P = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$, where $\mathbf{x}_i \in \mathbb{R}^d, 1 \le i \le n$, respectively, $S = (\mathbf{s}_1, \ldots, \mathbf{s}_{n_b})$, where $\mathbf{s}_i \in P$. In addition, assume we are also given two vectors $\mathbf{V} = (V_1, \ldots, V_n)^t$, where V_i is an volume weight of $\mathbf{x}_i$ in $\mathcal{M}$, and $\mathbf{A} = (A_1, \ldots, A_{n_b})^t$, where A_i is an area weight of $\mathbf{s}_i$ in $\partial \mathcal{M}$. In this point cloud data $(P, S, \mathbf{V}, \mathbf{A})$, the integral equation (2.3) can be disretized as

(2.12)
$$\frac{1}{t} \sum_{\mathbf{x}_j \in P} R_t(\mathbf{x}_i, \mathbf{x}_j)(u_i - u_j) p_j V_j = 2 \sum_{\mathbf{s}_j \in S} \bar{R}_t(\mathbf{x}_i, \mathbf{s}_j) b_j p_j A_j + \sum_{\mathbf{x}_j \in P} \bar{R}_t(\mathbf{x}_i, \mathbf{x}_j) f_j V_j / p_j,$$

where $p_j = p(\mathbf{x}_j)$, $f_j = f(\mathbf{x}_j)$, $b_l = b(\mathbf{s}_l)$, $j = 1, \ldots, |P|$, $l = 1, \ldots, |S|$.

The other integral equations and corresponding eigenvalue problems can be discretized consequently.

Remark 2.1. The integral approximation (1.6) also holds if the parameter t depends on $\mathbf{x}$, i.e.,

(2.13)
$$-\int_{\mathcal{M}} \mathrm{div}(p^2(\mathbf{y}) \nabla u(\mathbf{y})) \frac{\bar{R}_t(\mathbf{x}, \mathbf{y})}{p(\mathbf{y})} d\mu_{\mathbf{y}} \approx \frac{1}{t(\mathbf{x})} \int_{\mathcal{M}} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t(\mathbf{x})}\right)(u(\mathbf{x}) - u(\mathbf{y})) p(\mathbf{y}) d\mu_{\mathbf{y}}$$

$$- 2 \int_{\partial \mathcal{M}} \frac{\partial u}{\partial \mathbf{n}}(\mathbf{y}) \bar{R}\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t(\mathbf{x})}\right) p(\mathbf{y}) d\tau_{\mathbf{y}}.$$

Based on above approximation, in the computation, we can choose t adaptive to the distribution of the points.

3. Convergence analysis. In this section, we analyze the convergence of the PIM for the isotropic elliptic equation. To make the theoretical analysis concise, we only consider the homogeneous Neumann boundary conditions,

(3.1)
$$\begin{cases} -\mathrm{div}(p^2(\mathbf{x}) \nabla u(\mathbf{x})) = f(\mathbf{x}), & \mathbf{x} \in \mathcal{M}, \\ \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = 0, & \mathbf{x} \in \partial \mathcal{M}. \end{cases}$$

The corresponding numerical scheme is

(3.2)
$$\frac{1}{t} \sum_{\mathbf{x}_j \in P} R_t(\mathbf{x}_i, \mathbf{x}_j)(u_i - u_j) p_j V_j = \sum_{\mathbf{x}_j \in P} \bar{R}_t(\mathbf{x}_i, \mathbf{x}_j) f_j V_j / p_j.$$

The analysis can be easily generalized to the nonhomogeneous boundary conditions. The convergence of Dirichlet problem can also be proved following a similar procedure as that in [34].

3.1. Main result. We will prove that the solution given by the PIM converges to the exact solution as the point cloud $(P, \mathbf{V})$ converges to the manifold $\mathcal{M}$. Before giving the result of the convergence, we need to clarify the meaning of the convergence of the point cloud $(P, \mathbf{V})$ to the manifold $\mathcal{M}$.

First, we introduce an index to measure the distance between the point cloud $(P, \mathbf{V})$ and the manifold $\mathcal{M}$, which is called the *integral accuracy index*, denoted as $h(P, \mathbf{V}, \mathcal{M})$.

DEFINITION 3.1 (integral accuracy index). *For the point cloud $(P, \mathbf{V})$ which samples the manifold $\mathcal{M}$, the integral accuracy index $h(P, \mathbf{V}, \mathcal{M})$ is defined as*

$$h(P, \mathbf{V}, \mathcal{M}) = \sup_{f \in C^1(\mathcal{M})} \frac{\left| \int_{\mathcal{M}} f(\mathbf{y}) d\mu_{\mathbf{y}} - \sum_{\mathbf{x}_i \in P} f(\mathbf{x}_i) V_i \right|}{|\mathrm{supp}(f)| \|f\|_{C^1(\mathcal{M})}},$$

where $\|f\|_{C^1(\mathcal{M})} = \|f\|_\infty + \|\nabla f\|_\infty$ and $|\mathrm{supp}(f)|$ is the volume of the support of f.

Using the definition of integrable index, we say that the point cloud $(P, \mathbf{V})$ converges to the manifold $\mathcal{M}$ if $h(P, \mathbf{V}, \mathcal{M}) \to 0$. In the convergence analysis, we consider the case that $h(P, \mathbf{V}, \mathcal{M})$ is small enough.

Remark 3.1. In some sense, $h(P, \mathbf{V}, \mathcal{M})$ is a measure of the density of the point cloud. If the point cloud is uniformly distributed on the manifold, from the central limit theorem, $h(P, \mathbf{V}, \mathcal{M}) \sim O(1/\sqrt{|P|})$, where $|P|$ is the number of point in P.

Remark 3.2. To consider the nonhomogeneous Neumann boundary condition or Dirichlet boundary condition, we have to also assume that $h(S, \mathbf{A}, \partial\mathcal{M}) \to 0$, where S is the point set sample the boundary $\partial\mathcal{M}$ and $\mathbf{A}$ is the corresponding volume weight on the boundary $\partial\mathcal{M}$.

To get the convergence, we also need some assumptions on the regularity of the submanifold $\mathcal{M}$ and the integral kernel function R.

Assumption 3.1.
- Smoothness of the manifold: $\mathcal{M}, \partial\mathcal{M}$ are both compact and C^∞ smooth k-dimensional submanifolds isometrically embedded in a Euclidean space $\mathbb{R}^d$.
- Ellipticity: there exist generic constants $c_0, c_1 > 0$, such that $c_0 \leq p(\mathbf{x}) \leq c_1$ and $p(\mathbf{x}) \in C^1(\mathcal{M})$.
- Assumptions on the kernel function $R(r)$:
  (a) smoothness: $R \in C^2(\mathbb{R}^+)$;
  (b) nonnegativity: $R(r) \geq 0$ for any $r \geq 0$.
  (c) compact support: $R(r) = 0$ for $\forall r > 1$;
  (d) nondegeneracy: $\exists \delta_0 > 0$ so that $R(r) \geq \delta_0$ for $0 \leq r \leq \frac{1}{2}$.

Remark 3.3. The assumption on the kernel function is very mild. The compact support assumption can be relaxed to exponentially decay, like a Gaussian kernel. In the nondegeneracy assumption, $1/2$ may be replaced by a positive number θ_0 with $0 < \theta_0 < 1$. Similar assumptions on the kernel function are also used in analysis the nonlocal diffusion problem [15].

All the convergence analysis in this paper is based on the above assumptions. In the statement of the theorems, the above assumptions are omitted to make the statements more concise.

The other issue we have to address is how to compute the difference between the discrete solution and the analytic solution. The solution of the discrete system (3.2) is a vector $\mathbf{u}$ defined on P while the solution of the problem (3.1) is a function defined on $\mathcal{M}$. To make them comparable, for any solution $\mathbf{u} = (u_1, \ldots, u_n)^t$, $n = |P|$ to the problem (3.2), we construct a function on $\mathcal{M}$

$$(3.3) \qquad I_{\mathbf{f}}(\mathbf{u})(\mathbf{x}) = \frac{\sum_{\mathbf{x}_j \in P} R_t(\mathbf{x}, \mathbf{x}_j) u_j p_j V_j + t \sum_{\mathbf{x}_j \in P} \bar{R}_t(\mathbf{x}, \mathbf{x}_j) f_j V_j / p_j}{\sum_{\mathbf{x}_j \in P} R_t(\mathbf{x}, \mathbf{x}_j) p_j V_j}.$$

It is easy to verify that $I_{\mathbf{f}}(\mathbf{u})$ interpolates $\mathbf{u}$ at the sample points P, i.e., $I_{\mathbf{f}}(\mathbf{u})(\mathbf{x}_j) = u_j$ for any $\mathbf{x}_j \in P$. The following theorem guarantees the convergence of the PIM.

THEOREM 3.2. *Let u be the solution to problem* (3.1) *with $f \in C^1(\mathcal{M})$ and let the vector $\mathbf{u}$ be the solution to the problem* (3.2). *Then there exists constants C and T_0 depending on $\mathcal{M}$ and $p(\mathbf{x})$, such that for any t, $\frac{h(P, \mathbf{V}, \mathcal{M})}{\sqrt{t}} \le T_0$,*

$$(3.4) \qquad \|u - I_{\mathbf{f}}(\mathbf{u})\|_{H^1(\mathcal{M})} \le C \left(t^{1/2} + \frac{h(P, \mathbf{V}, \mathcal{M})}{t^{3/2}} \right) \|f\|_{C^1(\mathcal{M})},$$

where $h(P, \mathbf{V}, \mathcal{M})$ is the integral accuracy index.

3.2. Proof of convergence. Roughly, the proof the convergence includes two parts: an estimate of the truncation error $L_t(u - I_{\mathbf{f}}(\mathbf{u}))$ and the stability of the integral operator L_t. Here L_t is the integral operator in (2.1), $u(\mathbf{x})$ is the solution of the problem (3.1), and $\mathbf{u}$ is the solution of the problem (3.2).

This strategy is standard in numerical analysis. It is well known that consistency together with stability implies convergence. On the other hand, the PIM has some special structures both in truncation error and stability, which makes the analysis a little more involved.

First, we have following theorem regarding the stability of the operator L_t.

THEOREM 3.3. *Let $u(\mathbf{x})$ solves the integral equation*

$$L_t u = r(\mathbf{x}),$$

where $r \in H^1(\mathcal{M})$ with $\int_{\mathcal{M}} r(\mathbf{x}) p(\mathbf{x}) \mathrm{d}\mu_{\mathbf{x}} = 0$. Then, there exist constants $C > 0, T_0 > 0$ independent on t, such that

$$\|u\|_{H^1(\mathcal{M})} \le C \left(\|r\|_{L^2(\mathcal{M})} + t \|\nabla r\|_{L^2(\mathcal{M})} \right)$$

as long as $t \le T_0$.

To use above stability result, we need L_2 estimate of $L_t(u - I_{\mathbf{f}}(\mathbf{u}))$ and $\nabla L_t(u - I_{\mathbf{f}}(\mathbf{u}))$. In the analysis, we split the truncation error $L_t(u - I_{\mathbf{f}}(\mathbf{u}))$ to two terms,

$$L_t(u - I_{\mathbf{f}}(\mathbf{u})) = L_t(u - u_t)) + L_t(u_t - I_{\mathbf{f}}(\mathbf{u})),$$

where u_t is the solution of the integral equation

$$(3.5) \qquad \frac{1}{t} \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y})(u(\mathbf{x}) - u(\mathbf{y})) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} = \int_{\mathcal{M}} f(\mathbf{y}) \frac{\bar{R}_t(\mathbf{x}, \mathbf{y})}{p(\mathbf{y})} \mathrm{d}\mu_{\mathbf{y}}.$$

For the second term, we have following estimate.

THEOREM 3.4. *Let $u_t(\mathbf{x})$ be the solution of the problem (3.5) and $\mathbf{u}$ be the solution of the problem (3.2). If $f \in C^1(\mathcal{M})$, then there exists constants C, T_0 depending only on $\mathcal{M}$ and the coefficient $p(\mathbf{x})$, so that*

$$(3.6) \qquad \|L_t (I_{\mathbf{f}}\mathbf{u} - u_t)\|_{L^2(\mathcal{M})} \leq \frac{Ch(P, \mathbf{V}, \mathcal{M})}{t^{3/2}} \|f\|_{C^1(\mathcal{M})},$$

$$(3.7) \qquad \|\nabla L_t (I_{\mathbf{f}}\mathbf{u} - u_t)\|_{L^2(\mathcal{M})} \leq \frac{Ch(P, \mathbf{V}, \mathcal{M})}{t^2} \|f\|_{C^1(\mathcal{M})},$$

as long as $t \leq T_0$ and $\frac{h(P, \mathbf{V}, \mathcal{M})}{\sqrt{t}} \leq T_0$, $h(P, \mathbf{V}, \mathcal{M})$ is the integral difference index in Definition 3.1.

The error term $L_t(u - u_t))$ is a little more complicated. It has two parts: one is the interior term and the other is the boundary term. We need to estimate these two terms separately to get better estimation of the convergence rate.

THEOREM 3.5. *Let $u(\mathbf{x})$ be the solution of the problem (3.1) and $u_t(\mathbf{x})$ be the solution of the corresponding integral equation (3.5). Let*

$$(3.8) \qquad I_{bd} = \sum_{j=1}^{d} \int_{\partial\mathcal{M}} n^j(\mathbf{y})(\mathbf{x} - \mathbf{y}) \cdot \nabla(\nabla^j u(\mathbf{y})) \bar{R}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}}$$

and

$$L_t(u - u_t) = I_{in} + I_{bd},$$

where $\mathbf{n}(\mathbf{y}) = (n^1(\mathbf{y}), \ldots, n^d(\mathbf{y}))$ is the out normal vector of $\partial\mathcal{M}$ at $\mathbf{y}$, and ∇^j is the jth component of gradient ∇.

If $u \in H^3(\mathcal{M})$, then there exists constants C, T_0 depending only on $\mathcal{M}$ and $p(\mathbf{x})$, so that

$$(3.9) \qquad \|I_{in}\|_{L^2(\mathcal{M})} \leq Ct^{1/2} \|u\|_{H^3(\mathcal{M})}, \quad \|\nabla I_{in}\|_{L^2(\mathcal{M})} \leq C \|u\|_{H^3(\mathcal{M})},$$

as long as $t \leq T_0$.

Using the definition of the boundary term I_{bd}, (3.8), it is easy to check that

$$\|I_{bd}\|_{L^2(\mathcal{M})} = O(t^{1/4}), \quad \|\nabla I_{bd}\|_{L^2(\mathcal{M})} = O(t^{-1/2}).$$

Based on this estimation, Theorems 3.3 and 3.5 give that

$$\|u - u_t\|_{H^1(\mathcal{M})} = O(t^{1/4}).$$

This proves the convergence, but the convergence rate is relatively low. This low rate comes from the boundary term. From the interior term only, the rate is $\sqrt{t}$. Notice that the boundary term has a specific integral formula given in (3.8). Using this formula, we know that the boundary term concentrates in a small layer adjacent to the boundary whose width is of the order of $\sqrt{t}$ and vanishes in the interior region. Utilizing this special structure, we could get a better convergence rate with the help of a stability estimate specifically for the boundary term, which is given in Theorem 3.6.

THEOREM 3.6. *Let $u(\mathbf{x})$ solve the integral equation*

$$L_t u = \int_{\partial\mathcal{M}} \mathbf{b}(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \bar{R}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}} - \bar{b},$$

where $|\mathcal{M}|_p = \int_{\mathcal{M}} p(\mathbf{x}) \mathrm{d}\mu_{\mathbf{x}}$ and

$$\bar{b} = \frac{1}{|\mathcal{M}|_p} \int_{\mathcal{M}} \left(\int_{\partial \mathcal{M}} \mathbf{b}(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \bar{R}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}} \right) \mathrm{d}\mathbf{x}.$$

Then, there exist constant $C > 0, T_0 > 0$ independent on t, such that

$$\|u\|_{H^1(\mathcal{M})} \le C\sqrt{t} \, \|\mathbf{b}\|_{H^1(\mathcal{M})},$$

as long as $t \le T_0$.

Based on above four theorems, it is easy to prove Theorem 3.2. Using Theorems 3.4 and 3.3, we get

$$\|u_t - I_{\mathbf{f}}(\mathbf{u})\|_{H^1(\mathcal{M})} = O\left(\frac{h(P, \mathbf{V}, \mathcal{M})}{t^{3/2}} \right).$$

Applying Theorem 3.3 to the interior terms in Theorems 3.5 and 3.6 to the boundary term, respectively, we have

$$\|u - u_t\|_{H^1(\mathcal{M})} = O\left(t^{1/2} \right).$$

Putting above two inequality together, Theorem 3.2 is proved.

Next, we prove Theorems 3.3, 3.4, 3.5, and 3.6, respectively.

3.3. Proof of Theorem 3.5. Let $r(\mathbf{x}) = L_t u - L_t u_t$, where u and u_t are the solution of (3.1) and (3.5), respectively. Using integration by parts, we have

(3.10)

$$\begin{aligned}
r(\mathbf{x}) &= \frac{1}{t} \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y})(u(\mathbf{x}) - u(\mathbf{y})) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} - \int_{\mathcal{M}} \mathrm{div}(p^2(\mathbf{y}) \nabla u(\mathbf{y})) \frac{\bar{R}_t(\mathbf{x}, \mathbf{y})}{p(\mathbf{y})} \mathrm{d}\mu_{\mathbf{y}} \\
&\quad - 2 \int_{\partial \mathcal{M}} \bar{R}_t(\mathbf{x}, \mathbf{y}) \frac{\partial u}{\partial \mathbf{n}}(\mathbf{y}) p(\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}} \\
&= \frac{1}{t} \int_{\mathcal{M}} (u(\mathbf{x}) - u(\mathbf{y}) - (\mathbf{x} - \mathbf{y}) \cdot \nabla u(\mathbf{y})) R_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} \\
&\quad - \int_{\mathcal{M}} \Delta_{\mathcal{M}} u(\mathbf{y}) \bar{R}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}}.
\end{aligned}$$

The main idea of the proof is the Taylor expansion,

$$u(\mathbf{x}) - u(\mathbf{y}) - (\mathbf{x} - \mathbf{y}) \cdot \nabla u(\mathbf{y}) = \frac{1}{2}(\mathbf{x} - \mathbf{y})^T \cdot \mathbf{H}_u(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) + O(|\mathbf{x} - \mathbf{y}|^3),$$

where $\mathbf{H}_u(\mathbf{y})$ is the Hessian matrix of u at $\mathbf{y}$.

Using integration by parts, the second order term actually gives a Laplace–Beltrami operator which cancels with the second term in (3.10).

In manifold, the Taylor expansion and integration by parts are more complicated. To make the whole idea rigorous, we need to introduce a special parametrization of the manifold $\mathcal{M}$. This parametrization is based on following proposition.

PROPOSITION 3.7. *Assume both $\mathcal{M}$ and $\partial \mathcal{M}$ are C^2 smooth and σ is the minimum of the reaches of $\mathcal{M}$ and $\partial \mathcal{M}$. For any point $\mathbf{x} \in \mathcal{M}$, there is a neighborhood*

$U \subset \mathcal{M}$ of $\mathbf{x}$, so that there is a parametrization $\Phi : \Omega \subset \mathbb{R}^k \to U$ satisfying the following conditions. For any $\rho \leq 0.1$,

(i) Ω is convex and contains at least half of the ball $B_{\Phi^{-1}(\mathbf{x})}(\frac{\rho}{5}\sigma)$, i.e., $vol(\Omega \cap B_{\Phi^{-1}(\mathbf{x})}(\frac{\rho}{5}\sigma)) > \frac{1}{2}(\frac{\rho}{5}\sigma)^k w_k$, where w_k is the volume of unit ball in $\mathbb{R}^k$;

(ii) $B_{\mathbf{x}}(\frac{\rho}{10}\sigma) \cap \mathcal{M} \subset U$;

(iii) the determinant of the Jacobian of Φ is bounded; $(1-2\rho)^k \leq |D\Phi| \leq (1+2\rho)^k$ over Ω;

(iv) for any points $\mathbf{y}, \mathbf{z} \in U$, $1 - 2\rho \leq \frac{|\mathbf{y}-\mathbf{z}|}{|\Phi^{-1}(\mathbf{y})-\Phi^{-1}(\mathbf{z})|} \leq 1 + 3\rho$.

This proposition basically says there exists a local parametrization of small distortion if $(\mathcal{M}, \partial\mathcal{M})$ satisfies certain smoothness, and moreover, the parameter domain is convex and big enough. The proof of this proposition can be found in [33], and for the sake of completeness, we give the proof in the supplementary material. Next, we introduce a special parametrization of the manifold $\mathcal{M}$.

Let $\rho = 0.1$, σ be the minimum of the reaches of $\mathcal{M}$ and $\partial\mathcal{M}$ and $\delta = \rho\sigma/20$. For any $\mathbf{x} \in \mathcal{M}$, denote

$$(3.11) \qquad B_{\mathbf{x}}^\delta = \{\mathbf{y} \in \mathcal{M} : |\mathbf{x} - \mathbf{y}| \leq \delta\}, \quad \mathcal{M}_{\mathbf{x}}^t = \{\mathbf{y} \in \mathcal{M} : |\mathbf{x} - \mathbf{y}|^2 \leq 4t\},$$

and we assume t is small enough such that $2\sqrt{t} \leq \delta$.

Since the manifold $\mathcal{M}$ is compact, there exists a δ-net, $\mathcal{N}_\delta = \{\mathbf{q}_i \in \mathcal{M}, \ i = 1, \ldots, N\}$, such that

$$\mathcal{M} \subset \bigcup_{i=1}^N B_{\mathbf{q}_i}^\delta,$$

and there exists a partition of $\mathcal{M}$, $\{\mathcal{O}_i, \ i = 1, \ldots, N\}$, such that $\mathcal{O}_i \cap \mathcal{O}_j = \emptyset, \ i \neq j$ and

$$\mathcal{M} = \bigcup_{i=1}^N \mathcal{O}_i, \quad \mathcal{O}_i \subset B_{\mathbf{q}_i}^\delta, \quad i = 1, \ldots, N.$$

Using Proposition 3.7, there exists a parametrization $\Phi_i : \Omega_i \subset \mathbb{R}^k \to U_i \subset \mathcal{M}, \ i = 1, \ldots, N$, such that

1. (convexity) $B_{\mathbf{q}_i}^{2\delta} \subset U_i$ and Ω_i is convex;
2. (smoothness) $\Phi_i \in C^3(\Omega_i)$;
3. (locally small deformation) for any points $\theta_1, \theta_2 \in \Omega_i$,

$$\frac{1}{2}|\theta_1 - \theta_2| \leq \|\Phi_i(\theta_1) - \Phi_i(\theta_2)\| \leq 2|\theta_1 - \theta_2|.$$

Using the partition, $\{\mathcal{O}_i, \ i = 1, \ldots, N\}$, for any $\mathbf{y} \in \mathcal{M}$, there exists unique $J(\mathbf{y}) \in \{1, \ldots, N\}$, such that

$$(3.12) \qquad \mathbf{y} \in \mathcal{O}_{J(\mathbf{y})} \subset B_{\mathbf{q}_{J(\mathbf{y})}}^\delta.$$

Moerover, using the condition, $2\sqrt{t} \leq \delta$, we have $\mathcal{M}_{\mathbf{y}}^t \subset B_{\mathbf{q}_{J(\mathbf{y})}}^{2\delta} \subset U_{J(\mathbf{y})}$. Then $\Phi_{J(\mathbf{y})}^{-1}(\mathbf{x})$ and $\Phi_{J(\mathbf{y})}^{-1}(\mathbf{y})$ are both well defined for any $\mathbf{x} \in \mathcal{M}_{\mathbf{y}}^t$.

Now, we define an auxiliary function, $\eta(\mathbf{x}, \mathbf{y})$ for any $\mathbf{y} \in \mathcal{M}$, $\mathbf{x} \in \mathcal{M}_{\mathbf{y}}^t$. Let

(3.13)
$$\xi(\mathbf{x}, \mathbf{y}) = \Phi_{J(\mathbf{y})}^{-1}(\mathbf{x}) - \Phi_{J(\mathbf{y})}^{-1}(\mathbf{y}) \in \mathbb{R}^k, \quad \eta(\mathbf{x}, \mathbf{y}) = \xi(\mathbf{x}, \mathbf{y}) \cdot \partial \Phi_{J(\mathbf{y})}(\alpha(\mathbf{x}, \mathbf{y})) \in \mathbb{R}^d,$$

where $\alpha(\mathbf{x}, \mathbf{y}) = \Phi_{J(\mathbf{y})}^{-1}(\mathbf{y})$ and ∂ is the gradient operator in the parameter space, i.e.,

$$\partial \Phi_j(\theta) = \left(\frac{\partial \Phi_j}{\partial \theta_1}(\theta), \frac{\partial \Phi_j}{\partial \theta_2}(\theta), \ldots, \frac{\partial \Phi_j}{\partial \theta_k}(\theta) \right), \quad \theta \in \Omega_j \subset \mathbb{R}^k.$$

Now we state the proof of Theorem 3.5.

Proof. First, we split the residual $r(\mathbf{x})$ in (3.10) to four terms

$$r(\mathbf{x}) = r_1(\mathbf{x}) + r_2(\mathbf{x}) + r_3(\mathbf{x}) - r_4(\mathbf{x}),$$

where

$$r_1(\mathbf{x}) = \frac{1}{t} \int_{\mathcal{M}} \left(u(\mathbf{x}) - u(\mathbf{y}) - (\mathbf{x} - \mathbf{y}) \cdot \nabla u(\mathbf{y}) - \frac{1}{2} \eta^i \eta^j (\nabla^i \nabla^j u(\mathbf{y})) \right) R_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}},$$

$$r_2(\mathbf{x}) = \frac{1}{2t} \int_{\mathcal{M}} \eta^i \eta^j (\nabla^i \nabla^j u(\mathbf{y})) R_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} - \int_{\mathcal{M}} \eta^i (\nabla^i \nabla^j u(\mathbf{y}) \nabla^j \bar{R}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}},$$

$$r_3(\mathbf{x}) = \int_{\mathcal{M}} \eta^i (\nabla^i \nabla^j u(\mathbf{y}) \nabla^j \bar{R}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} + \int_{\mathcal{M}} \mathrm{div} \left(\eta^i (\nabla^i \nabla u(\mathbf{y})) \bar{R}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}}, \right.$$

$$r_4(\mathbf{x}) = \int_{\mathcal{M}} \mathrm{div} \left(\eta^i (\nabla^i \nabla u(\mathbf{y})) \bar{R}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} + \int_{\mathcal{M}} \Delta_{\mathcal{M}} u(\mathbf{y}) \bar{R}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}}, \right.$$

where ∇^i, $i = 1, \ldots, d$, is the ith component of the gradient ∇, and η^i, $i = 1, \ldots, d$, is the ith component of $\eta(\mathbf{x}, \mathbf{y})$ defined in (3.13). To simplify the notation, we drop the variable $(\mathbf{x}, \mathbf{y})$ in the function $\eta(\mathbf{x}, \mathbf{y})$.

Next, we will prove the theorem by estimating the above four terms one by one. First, we consider r_1. Letting

$$d(\mathbf{x}, \mathbf{y}) = u(\mathbf{x}) - u(\mathbf{y}) - (\mathbf{x} - \mathbf{y}) \cdot \nabla u(\mathbf{y}) - \frac{1}{2} \eta^i \eta^j (\nabla^i \nabla^j u(\mathbf{y})),$$

we have

$$\int_{\mathcal{M}} |r_1(\mathbf{x})|^2 \mathrm{d}\mu_{\mathbf{x}} = \int_{\mathcal{M}} \left| \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) d(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} \right|^2 \mathrm{d}\mu_{\mathbf{x}}$$

$$\leq (\max_{\mathbf{y}} p(\mathbf{y}))^2 \int_{\mathcal{M}} \left(\int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} \right) \left(\int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) |d(\mathbf{x}, \mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{y}} \right) \mathrm{d}\mu_{\mathbf{x}}$$

$$\leq C \int_{\mathcal{M}} \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) |d(\mathbf{x}, \mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{x}}$$

and

$$\int_{\mathcal{M}} \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) |d(\mathbf{x}, \mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{x}} = \sum_{i=1}^{N} \int_{\mathcal{M}} \int_{\mathcal{O}_i} R_t(\mathbf{x}, \mathbf{y}) |d(\mathbf{x}, \mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{x}}$$

$$= \sum_{i=1}^{N} \int_{\mathcal{O}_i} \left(\int_{\mathcal{M}_{\mathbf{y}}^t} R_t(\mathbf{x}, \mathbf{y}) |d(\mathbf{x}, \mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{x}} \right) \mathrm{d}\mu_{\mathbf{y}}.$$

Using the Newton–Leibniz formula, we get

$$
d(\mathbf{x}, \mathbf{y})
$$
$$
= u(\mathbf{x}) - u(\mathbf{y}) - (\mathbf{x} - \mathbf{y}) \cdot \nabla u(\mathbf{y}) - \frac{1}{2}\eta^i \eta^j (\nabla^i \nabla^j u(\mathbf{y}))
$$
$$
= \xi^i \xi^{i'} \int_0^1 \int_0^1 \int_0^1 s_1 \frac{d}{ds_3}\Big(\partial_i \Phi^j(\alpha + s_3 s_1 \xi)\partial_{i'}\Phi^{j'}(\alpha + s_3 s_2 s_1 \xi)
$$
$$
\nabla^{j'}\nabla^j u(\Phi(\alpha + s_3 s_2 s_1 \xi))\Big)\mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1
$$
$$
= \xi^i \xi^{i'}\xi^{i''} \int_0^1 \int_0^1 \int_0^1 s_1^2 s_2 \partial_i \Phi^j(\alpha + s_3 s_1 \xi)\partial_{i''}\partial_{i'}\Phi^{j'}(\alpha + s_3 s_2 s_1 \xi)
$$
$$
\nabla^{j'}\nabla^j u(\Phi(\alpha + s_3 s_2 s_1 \xi))\mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1
$$
$$
+ \xi^i \xi^{i'}\xi^{i''} \int_0^1 \int_0^1 \int_0^1 s_1^2 \partial_{i''}\partial_i \Phi^j(\alpha + s_3 s_1 \xi)\partial_{i'}\Phi^{j'}(\alpha + s_3 s_2 s_1 \xi)
$$
$$
\nabla^{j'}\nabla^j u(\Phi(\alpha + s_3 s_2 s_1 \xi))\mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1
$$
$$
+ \xi^i \xi^{i'}\xi^{i''} \int_0^1 \int_0^1 \int_0^1 s_1^2 s_2 \partial_i \Phi^j(\alpha + s_3 s_2 s_1 \xi)\partial_{i'}\Phi^{j'}(\alpha + s_3 s_2 s_1 \xi)
$$
$$
\partial_{i''}\Phi^{j''}(\alpha + s_3 s_2 s_1 \xi)\nabla^{j''}\nabla^{j'}\nabla^j u(\Phi(\alpha + s_3 s_2 s_1 \xi))\mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1.
$$

Here, Φ^i, $i = 1, \ldots, d$ is the ith component of the parameterization function Φ and the parameterization function $\Phi = \Phi_{J(\mathbf{y})}$, and $J(\mathbf{y})$ is the index function given in (3.12). $\alpha = \alpha(\mathbf{x}, \mathbf{y}) = \Phi^{-1}_{J(\mathbf{y})}(\mathbf{y})$, $\xi = \xi(\mathbf{x}, \mathbf{y}) = \Phi^{-1}_{J(\mathbf{y})}(\mathbf{x}) - \Phi^{-1}_{J(\mathbf{y})}(\mathbf{y})$. In the rest of the proof, without introducing any confusion, we always use this short notation to save space. In the above derivation, we need the convexity property of the parameterization function to make sure all the integrals are well defined.

Using the above equality and the smoothness of the parameterization functions, it is easy to show that

$$
\int_{\mathcal{O}_i} \left(\int_{\mathcal{M}_{\mathbf{y}}^t} R_t(\mathbf{x}, \mathbf{y})|d(\mathbf{x}, \mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{x}} \right) \mathrm{d}\mu_{\mathbf{y}}
$$
$$
\leq Ct^3 \int_0^1 \int_0^1 \int_0^1 \int_{\mathcal{O}_i} \int_{\mathcal{M}_{\mathbf{y}}^t} R_t(\mathbf{x}, \mathbf{y}) \left| D^{2,3}u(\Phi_{J(\mathbf{y})}(\alpha + s_3 s_2 s_1 \xi)) \right|^2 \mathrm{d}\mu_{\mathbf{x}}\mathrm{d}\mu_{\mathbf{y}}\mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1
$$
$$
\leq Ct^3 \max_{0 \leq s \leq 1} \int_{\mathcal{O}_i} \int_{\mathcal{M}_{\mathbf{y}}^t} R_t(\mathbf{x}, \mathbf{y}) \left| D^{2,3}u(\Phi_i(\alpha + s\xi)) \right|^2 \mathrm{d}\mu_{\mathbf{x}}\mathrm{d}\mu_{\mathbf{y}},
$$

where we use the fact that $J(\mathbf{y}) = i$, $\mathbf{y} \in \mathcal{O}_i$ and

$$
\left| D^{2,3}u(\mathbf{x}) \right|^2 = \sum_{j,j',j''=1}^{d} |\nabla^{j''}\nabla^{j'}\nabla^j u(\mathbf{x})|^2 + \sum_{j,j'=1}^{d} |\nabla^{j'}\nabla^j u(\mathbf{x})|^2.
$$

Let $\mathbf{z}_i = \Phi_i(\alpha + s\xi)$, $0 \leq s \leq 1$, and then for any $\mathbf{y} \in \mathcal{O}_i \subset B_{\mathbf{q}_i}^\delta$ and $\mathbf{x} \in \mathcal{M}_{\mathbf{y}}^t$,

$$
|\mathbf{z}_i - \mathbf{y}| \leq 2s|\xi| \leq 4s|\mathbf{x} - \mathbf{y}| \leq 8s\sqrt{t}, \quad |\mathbf{z}_i - \mathbf{q}_i| \leq |\mathbf{z}_i - \mathbf{y}| + |\mathbf{y} - \mathbf{q}_i| \leq \delta + 8s\sqrt{t}.
$$

We can assume that t is small enough such that $8\sqrt{t} \leq \delta$, and then we have

$$
\mathbf{z}_i \in B_{\mathbf{q}_i}^{2\delta}.
$$

After a changing of variable, we obtain

$$
\int_{\mathcal{O}_i} \int_{\mathcal{M}_{\mathbf{y}}^t} R_t(\mathbf{x}, \mathbf{y}) \left| D^{2,3} u(\Phi_i(\alpha + s\xi)) \right|^2 \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\mu_{\mathbf{y}}
$$

$$
\leq \frac{C}{\delta_0} \int_{\mathcal{O}_i} \int_{B_{\mathbf{q}_i}^{2\delta}} \frac{1}{s^k} R\left(\frac{|\mathbf{z}_i - \mathbf{y}|^2}{128 s^2 t} \right) \left| D^{2,3} u(\mathbf{z}_i) \right|^2 \mathrm{d}\mu_{\mathbf{z}_i} \mathrm{d}\mu_{\mathbf{y}}
$$

$$
= \frac{C}{\delta_0} \int_{\mathcal{O}_i} \frac{1}{s^k} R\left(\frac{|\mathbf{z}_i - \mathbf{y}|^2}{128 s^2 t} \right) \mathrm{d}\mu_{\mathbf{y}} \int_{B_{\mathbf{q}_i}^{2\delta}} \left| D^{2,3} u(\mathbf{z}_i) \right|^2 \mathrm{d}\mu_{\mathbf{z}_i}
$$

$$
\leq C \int_{B_{\mathbf{q}_i}^{2\delta}} \left| D^{2,3} u(\mathbf{x}) \right|^2 \mathrm{d}\mu_{\mathbf{x}}.
$$

This estimate would give us that

$$
(3.14) \qquad \|r_1(\mathbf{x})\|_{L^2(\mathcal{M})} \leq C t^{1/2} \|u\|_{H^3(\mathcal{M})}.
$$

Now, we turn to estimate the gradient of r_1.

$$
\int_{\mathcal{M}} |\nabla_{\mathbf{x}} r_1(\mathbf{x})|^2 \mathrm{d}\mu_{\mathbf{x}} \leq C \int_{\mathcal{M}} \left| \int_{\mathcal{M}} \nabla_{\mathbf{x}} R_t(\mathbf{x}, \mathbf{y}) d(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} \right|^2 \mathrm{d}\mu_{\mathbf{x}}
$$

$$
+ C \int_{\mathcal{M}} \left| \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) \nabla_{\mathbf{x}} d(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} \right|^2 \mathrm{d}\mu_{\mathbf{x}},
$$

where $\nabla_{\mathbf{x}}$ is the gradient in $\mathcal{M}$ with respect to $\mathbf{x}$.

Using the same techniques in the calculation of $\|r_1(\mathbf{x})\|_{L^2(\mathcal{M})}$, we get that the first term of right-hand side can bounded as

$$
\int_{\mathcal{M}} \left| \int_{\mathcal{M}} \nabla_{\mathbf{x}} R_t(\mathbf{x}, \mathbf{y}) d(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} \right|^2 \mathrm{d}\mu_{\mathbf{x}} \leq C \|u\|_{H^3(\mathcal{M})}^2.
$$

The estimation of second term is a little involved. First, we have

$$
\int_{\mathcal{M}} \left| \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) \nabla_{\mathbf{x}} d(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} \right|^2 \mathrm{d}\mu_{\mathbf{x}}
$$

$$
\leq C \int_{\mathcal{M}} \left(\int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) |\nabla_{\mathbf{x}} d(\mathbf{x}, \mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{y}} \right) \mathrm{d}\mu_{\mathbf{x}}
$$

$$
= C \sum_{i=1}^{N} \int_{\mathcal{O}_i} \left(\int_{\mathcal{M}_{\mathbf{y}}^t} R_t(\mathbf{x}, \mathbf{y}) |\nabla_{\mathbf{x}} d(\mathbf{x}, \mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{x}} \right) \mathrm{d}\mu_{\mathbf{y}}.
$$

Also using the Newton–Leibniz formula, we have

$$
d(\mathbf{x}, \mathbf{y})
$$

$$
= \xi^i \xi^{i'} \int_0^1 \int_0^1 s_1 \left(\partial_i \Phi^j(\alpha + s_1 \xi) \partial_{i'} \Phi^{j'}(\alpha + s_2 s_1 \xi) \nabla^{j'} \nabla^j u(\Phi(\alpha + s_2 s_1 \xi)) \right) \mathrm{d}s_2 \mathrm{d}s_1
$$

$$
- \xi^i \xi^{i'} \int_0^1 \int_0^1 s_1 \left(\partial_i \Phi^j(\alpha) \partial_{i'} \Phi^{j'}(\alpha) \nabla^{j'} \nabla^j u(\Phi(\alpha)) \right) \mathrm{d}s_2 \mathrm{d}s_1.
$$

Then the gradient of $d(\mathbf{x}, \mathbf{y})$ has the following representation:

$$\nabla_{\mathbf{x}} d(\mathbf{x}, \mathbf{y}) = \xi^i \xi^{i'} \nabla_{\mathbf{x}} \left(\int_0^1 \int_0^1 s_1 \Big(\partial_i \Phi^j(\alpha + s_1 \xi) \partial_{i'} \Phi^{j'}(\alpha + s_2 s_1 \xi) \right.$$
$$\left. \nabla^{j'} \nabla^j u(\Phi(\alpha + s_2 s_1 \xi)) \Big) \mathrm{d}s_2 \mathrm{d}s_1 \right)$$
$$+ \nabla_{\mathbf{x}} \left(\xi^i \xi^{i'} \right) \int_0^1 \int_0^1 \int_0^1 s_1 \frac{d}{ds_3} \Big(\partial_i \Phi^j(\alpha + s_3 s_1 \xi) \partial_{i'} \Phi^{j'}(\alpha + s_3 s_2 s_1 \xi)$$
$$\nabla^{j'} \nabla^j u(\Phi(\alpha + s_3 s_2 s_1 \xi)) \Big) \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1$$
$$= d_1(\mathbf{x}, \mathbf{y}) + d_2(\mathbf{x}, \mathbf{y}).$$

For d_1, we have

$$\int_{\mathcal{O}_i} \left(\int_{\mathcal{M}_{\mathbf{y}}^t} R_t(\mathbf{x}, \mathbf{y}) |d_1(\mathbf{x}, \mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{x}} \right) \mathrm{d}\mu_{\mathbf{y}}$$
$$\leq C t^2 \max_{0 \leq s \leq 1} \int_{\mathcal{O}_i} \left(\int_{\mathcal{M}_{\mathbf{y}}^t} R_t(\mathbf{x}, \mathbf{y}) |D^{2,3} u(\Phi_i(\alpha + s\xi))|^2 \mathrm{d}\mu_{\mathbf{x}} \right) \mathrm{d}\mu_{\mathbf{y}},$$

which means that

$$(3.15) \qquad \int_{\mathcal{O}_i} \left(\int_{\mathcal{M}_{\mathbf{y}}^t} R_t(\mathbf{x}, \mathbf{y}) |d_1(\mathbf{x}, \mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{x}} \right) \mathrm{d}\mu_{\mathbf{y}} \leq C \int_{B_{\mathbf{q}_i}^{2\delta}} |D^{2,3} u(\mathbf{x})|^2 \mathrm{d}\mu_{\mathbf{x}}.$$

For d_2, we have

$$d_2(\mathbf{x}, \mathbf{y})$$
$$= \nabla_{\mathbf{x}} \left(\xi^i \xi^{i'} \right) \int_{[0,1]^3} s_1 \frac{d}{ds_3} \Big(\partial_i \Phi^j(\alpha + s_3 s_1 \xi) \partial_{i'} \Phi^{j'}(\alpha + s_3 s_2 s_1 \xi)$$
$$\nabla^{j'} \nabla^j u(\Phi(\alpha + s_3 s_2 s_1 \xi)) \Big) \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1$$
$$= \nabla_{\mathbf{x}} \left(\xi^i \xi^{i'} \right) \xi^{i''} \int_{[0,1]^3} s_1^2 s_2 \partial_i \Phi^j(\alpha + s_3 s_1 \xi) \partial_{i''} \partial_{i'} \Phi^{j'}(\alpha + s_3 s_2 s_1 \xi)$$
$$\nabla^{j'} \nabla^j u(\Phi(\alpha + s_3 s_2 s_1 \xi)) \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1$$
$$+ \nabla_{\mathbf{x}} \left(\xi^i \xi^{i'} \right) \xi^{i''} \int_{[0,1]^3} s_1^2 \partial_{i''} \partial_i \Phi^j(\alpha + s_3 s_1 \xi) \partial_{i'} \Phi^{j'}(\alpha + s_3 s_2 s_1 \xi)$$
$$\nabla^{j'} \nabla^j u(\Phi(\alpha + s_3 s_2 s_1 \xi)) \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1$$
$$+ \nabla_{\mathbf{x}} \left(\xi^i \xi^{i'} \right) \xi^{i''} \int_{[0,1]^3} s_1^2 s_2 \partial_i \Phi^j(\alpha + s_2 s_1 \xi) \partial_{i'} \Phi^{j'}(\alpha + s_3 s_2 s_1 \xi) \partial_{i''} \Phi^{j''}(\alpha + s_3 s_2 s_1 \xi)$$
$$\nabla^{j''} \nabla^{j'} \nabla^j u(\Phi(\alpha + s_3 s_2 s_1 \xi)) \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1.$$

This formula tells us that

$$\int_{\mathcal{O}_i} \left(\int_{\mathcal{M}_{\mathbf{y}}^t} R_t(\mathbf{x}, \mathbf{y}) |d_2(\mathbf{x}, \mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{x}} \right) \mathrm{d}\mu_{\mathbf{y}}$$
$$\leq C t^2 \max_{0 \leq s \leq 1} \int_{\mathcal{O}_i} \left(\int_{\mathcal{M}_{\mathbf{y}}^t} R_t(\mathbf{x}, \mathbf{y}) |D^{2,3} u(\Phi(\alpha + s\xi))|^2 \mathrm{d}\mu_{\mathbf{x}} \right) \mathrm{d}\mu_{\mathbf{y}}.$$

Using the same arguments as that in the calculation of $\|r_1\|_{L^2(\mathcal{M})}$, we have

$$(3.16) \qquad \int_{\mathcal{O}_i}\left(\int_{\mathcal{M}_{\mathbf{y}}^t}R_t(\mathbf{x},\mathbf{y})|d_2(\mathbf{x},\mathbf{y})|^2\mathrm{d}\mu_{\mathbf{x}}\right)\mathrm{d}\mu_{\mathbf{y}} \le C\int_{B_{\mathbf{q}_i}^{2\delta}}|D^3u(\mathbf{x})|^2\mathrm{d}\mu_{\mathbf{x}}.$$

Combining (3.15) and (3.16), we have

$$(3.17) \qquad \|\nabla r_1(\mathbf{x})\|_{L^2(\mathcal{M})} \le C\|u\|_{H^3(\mathcal{M})}.$$

For r_2, first, notice that

$$\nabla^j\bar{R}_t(\mathbf{x},\mathbf{y}) = \frac{1}{2t}\partial_{m'}\Phi^j(\alpha)g^{m'n'}\partial_{n'}\Phi^i(\alpha)(x^i-y^i)R_t(\mathbf{x},\mathbf{y}),$$

$$\frac{\eta^j}{2t}R_t(\mathbf{x},\mathbf{y}) = \frac{1}{2t}\partial_{m'}\Phi^j(\alpha)g^{m'n'}\partial_{n'}\Phi^i(\alpha)\xi^{i'}\partial_{i'}\Phi^iR_t(\mathbf{x},\mathbf{y}).$$

Then, we have

$$\nabla^j\bar{R}_t(\mathbf{x},\mathbf{y}) - \frac{\eta^j}{2t}R_t(\mathbf{x},\mathbf{y})$$
$$= \frac{1}{2t}\partial_{m'}\Phi^i g^{m'n'}\partial_{n'}\Phi^j\left(x^j-y^j-\xi^{i'}\partial_{i'}\Phi^j\right)R_t(\mathbf{x},\mathbf{y})$$
$$= \frac{1}{2t}\xi^{i'}\xi^{j'}\partial_{m'}\Phi^i g^{m'n'}\partial_{n'}\Phi^j\left(\int_0^1\int_0^1 s\partial_{j'}\partial_{i'}\Phi^j(\alpha+\tau s\xi)\mathrm{d}\tau\mathrm{d}s\right)R_t(\mathbf{x},\mathbf{y}).$$

Thus, we get

$$\left|\nabla^j\bar{R}_t(\mathbf{x},\mathbf{y}) - \frac{\eta^j}{2t}R_t(\mathbf{x},\mathbf{y})\right| \le \frac{C|\xi|^2}{t}R_t(\mathbf{x},\mathbf{y}),$$

$$\left|\nabla_{\mathbf{x}}\left(\nabla^j\bar{R}_t(\mathbf{x},\mathbf{y}) - \frac{\eta^j}{2t}R_t(\mathbf{x},\mathbf{y})\right)\right| \le \frac{C|\xi|}{t}R_t(\mathbf{x},\mathbf{y}) + \frac{C|\xi|^3}{t^2}|R_t'(\mathbf{x},\mathbf{y})|.$$

Then, we have following bound for r_2:

$$(3.18) \qquad \int_{\mathcal{M}}|r_2(\mathbf{x})|^2\mathrm{d}\mu_{\mathbf{x}}$$
$$\le Ct\int_{\mathcal{M}}\left(\int_{\mathcal{M}}R_t(\mathbf{x},\mathbf{y})|D^2u(\mathbf{y})|p(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\right)^2\mathrm{d}\mu_{\mathbf{x}}$$
$$\le Ct\int_{\mathcal{M}}\left(\int_{\mathcal{M}}R_t(\mathbf{x},\mathbf{y})p(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\right)\int_{\mathcal{M}}R_t(\mathbf{x},\mathbf{y})|D^2u(\mathbf{y})|^2p(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{x}}$$
$$\le Ct\max_{\mathbf{y}}\left(\int_{\mathcal{M}}R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{x}}\right)\int_{\mathcal{M}}|D^2u(\mathbf{y})|^2p(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}$$
$$\le Ct\|u\|_{H^2(\mathcal{M})}^2.$$

Similarly, we have

$$(3.19) \quad \int_M|\nabla r_2(\mathbf{x})|^2\mathrm{d}\mu_{\mathbf{x}}$$
$$\le Ct\int_{\mathcal{M}}\left(\int_{\mathcal{M}}\nabla_{\mathbf{x}}R_t(\mathbf{x},\mathbf{y})p(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\right)\int_{\mathcal{M}}\nabla_{\mathbf{x}}R_t(\mathbf{x},\mathbf{y})|D^2u(\mathbf{y})|^2p(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{x}}$$
$$\le C\sqrt{t}\max_{\mathbf{y}}\left(\int_{\mathcal{M}}\nabla_{\mathbf{x}}R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{x}}\right)\int_{\mathcal{M}}|D^2u(\mathbf{y})|^2p(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}$$
$$\le C\|u\|_{H^2(\mathcal{M})}^2.$$

r_3 is relatively easy to estimate by using the well known Gauss formula.

$$r_3(\mathbf{x}) = \int_{\partial\mathcal{M}} n^j \eta^i (\nabla^i \nabla^j u(\mathbf{y})) \bar{R}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\tau_\mathbf{y} - \int_\mathcal{M} \eta^i (\nabla^i \nabla^j u(\mathbf{y})) \bar{R}_t(\mathbf{x}, \mathbf{y}) \nabla^j p(\mathbf{y}) \mathrm{d}\mu_\mathbf{y}$$

$$= \tilde{I}_{bd} - \int_\mathcal{M} \eta^i (\nabla^i \nabla^j u(\mathbf{y})) \bar{R}_t(\mathbf{x}, \mathbf{y}) \nabla^j p(\mathbf{y}) \mathrm{d}\mu_\mathbf{y},$$

where $\tilde{I}_{bd} = \int_{\partial\mathcal{M}} n^j \eta^i (\nabla^i \nabla^j u(\mathbf{y})) \bar{R}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\tau_\mathbf{y}$.

Using the assumption that $p \in C^1(\mathcal{M})$, it is easy to get that

$$(3.20) \qquad \|r_3 - \tilde{I}_{bd}\|_{L^2(\mathcal{M})} \leq C\sqrt{t} \|u\|_{H^2(\mathcal{M})},$$

$$(3.21) \qquad \|\nabla(r_3 - \tilde{I}_{bd})\|_{L^2(\mathcal{M})} \leq C\|u\|_{H^2(\mathcal{M})}.$$

Now, we turn to bound the last term r_4. Notice that

$$(3.22) \quad \nabla^j \left(\nabla^j u(\mathbf{y})\right) = (\partial_{k'} \Phi^j) g^{k'l'} \partial_{l'} \left((\partial_{m'} \Phi^j) g^{m'n'} (\partial_{n'} u)\right)$$

$$= (\partial_{k'} \Phi^j) g^{k'l'} \left(\partial_{l'} (\partial_{m'} \Phi^j)\right) g^{m'n'} (\partial_{n'} u)$$

$$+ (\partial_{k'} \Phi^j) g^{k'l'} (\partial_{m'} \Phi^j) \partial_{l'} \left(g^{m'n'} (\partial_{n'} u)\right)$$

$$= \frac{1}{\sqrt{\det G}} (\partial_{m'} \sqrt{\det G}) g^{m'n'} (\partial_{n'} u) + \partial_{m'} \left(g^{m'n'} (\partial_{n'} u)\right)$$

$$= \frac{1}{\sqrt{\det G}} \partial_{m'} \left(\sqrt{\det G} g^{m'n'} (\partial_{n'} u)\right) = \Delta_\mathcal{M} u(\mathbf{y}),$$

where $\det G$ is the determinant of G and $G = (g_{ij})_{i,j=1,\ldots,k}$. Here we use the fact that

$$(\partial_{k'} \Phi^j) g^{k'l'} \left(\partial_{l'} (\partial_{m'} \Phi^j)\right) = (\partial_{k'} \Phi^j) g^{k'l'} \left(\partial_{m'} (\partial_{l'} \Phi^j)\right)$$

$$= (\partial_{m'} (\partial_{k'} \Phi^j)) g^{k'l'} (\partial_{l'} \Phi^j)$$

$$= \frac{1}{2} g^{k'l'} \partial_{m'} (g_{k'l'})$$

$$= \frac{1}{\sqrt{\det G}} (\partial_{m'} \sqrt{\det G}).$$

Moreover, we have

$$(3.23) \qquad g^{i'j'} (\partial_{j'} \Phi^j)(\partial_{i'} \xi^l)(\partial_l \Phi^i)(\nabla^i \nabla^j u(\mathbf{y}))$$

$$= -g^{i'j'} (\partial_{j'} \Phi^j)(\partial_{i'} \Phi^i)(\nabla^i \nabla^j u(\mathbf{y}))$$

$$= -g^{i'j'} (\partial_{j'} \Phi^j)(\partial_{i'} \Phi^i)(\partial_{m'} \Phi^i) g^{m'n'} \partial_{n'} \left(\nabla^j u(\mathbf{y})\right)$$

$$= -g^{i'j'} (\partial_{j'} \Phi^j) \partial_{i'} \left(\nabla^j u(\mathbf{y})\right)$$

$$= -\nabla^j \left(\nabla^j u(\mathbf{y})\right),$$

where the first equalities are due to the fact that $\partial_{i'} \xi^l = -\delta_{i'}^l$. Then we have

$$\mathrm{div} \left(\eta^i (\nabla^i \nabla^j u(\mathbf{y}))\right) + \Delta_\mathcal{M} u(\mathbf{y})$$

$$= \frac{1}{\sqrt{\det G}} \partial_{i'} \left(\sqrt{\det G} \, g^{i'j'} (\partial_{j'} \Phi^j) \xi^l (\partial_l \Phi^i)(\nabla^i \nabla^j u(\mathbf{y}))\right)$$

$$- g^{i'j'} (\partial_{j'} \Phi^j)(\partial_{i'} \xi^l)(\partial_l \Phi^i)(\nabla^i \nabla^j u(\mathbf{y}))$$

$$= \frac{\xi^l}{\sqrt{\det G}} \partial_{i'} \left(\sqrt{\det G} \, g^{i'j'} (\partial_{j'} \Phi^j)(\partial_l \Phi^i)(\nabla^i \nabla^j u(\mathbf{y}))\right).$$

Here we use the equalities (3.22), (3.23), $\eta^i = \xi^l \partial_{i'} \Phi^l$ and the definition of div,

$$(3.24) \qquad \text{div} X = \frac{1}{\sqrt{\det G}} \partial_{i'} \left(\sqrt{\det G} \, g^{i'j'} \partial_{j'} \Phi^k X^k \right),$$

where X is a smooth tangent vector field on $\mathcal{M}$ and $(X^1, \ldots, X^d)^t$ is its representation in embedding coordinates.

Hence,

$$r_4(\mathbf{x}) = \int_{\mathcal{M}} \frac{\xi^l}{\sqrt{\det G}} \partial_{i'} \left(\sqrt{\det G} \, g^{i'j'} (\partial_{j'} \Phi^j)(\partial_l \Phi^i)(\nabla^i \nabla^j u(\mathbf{y})) \right) \bar{R}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}}.$$

Then it is easy to get that

$$(3.25) \qquad \|r_4(\mathbf{x})\|_{L^2(\mathcal{M})} \le C t^{1/2} \|u\|_{H^3(\mathcal{M})},$$
$$(3.26) \qquad \|\nabla r_4(\mathbf{x})\|_{L^2(\mathcal{M})} \le C \|u\|_{H^3(\mathcal{M})}.$$

By combining (3.14), (3.17), (3.18), (3.19), (3.20), (3.21), (3.25), (3.26), we know that

$$(3.27) \qquad \|r - \tilde{I}_{bd}\|_{L^2(\mathcal{M})} \le C t^{1/2} \|u\|_{H^3(\mathcal{M})},$$
$$(3.28) \qquad \|\nabla(r - \tilde{I}_{bd})\|_{L^2(\mathcal{M})} \le C \|u\|_{H^3(\mathcal{M})}.$$

Using the definition of I_{bd} and $\tilde{I}_{bd}$, we obtain

$$I_{bd} - \tilde{I}_{bd} = \int_{\partial \mathcal{M}} n^j(\mathbf{y})(\mathbf{x} - \mathbf{y} - \eta(\mathbf{x}, \mathbf{y})) \cdot (\nabla \nabla^j u(\mathbf{y})) \bar{R}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}}.$$

Using the definition of $\eta(\mathbf{x}, \mathbf{y})$, it is easy to check that

$$|\mathbf{x} - \mathbf{y} - \eta(\mathbf{x}, \mathbf{y})| = O(|\mathbf{x} - \mathbf{y}|^2), \quad |\nabla_{\mathbf{x}}(\mathbf{x} - \mathbf{y} - \eta(\mathbf{x}, \mathbf{y}))| = O(|\mathbf{x} - \mathbf{y}|),$$

which implies that

$$(3.29) \qquad \|I_{bd} - \tilde{I}_{bd}\|_{L^2(\mathcal{M})} \le C t^{3/4} \|u\|_{H^2(\mathcal{M})},$$
$$(3.30) \qquad \|\nabla(I_{bd} - \tilde{I}_{bd})\|_{L^2(\mathcal{M})} \le C t^{1/4} \|u\|_{H^3(\mathcal{M})}.$$

The theorem is proved by putting (3.27), (3.28), (3.29), (3.30) together. $\quad\square$

Remark 3.4. Using the above proof, we can also show that the L_2 error in the integral approximation (2.13) is $O(t^{1/4})$.

3.4. Proof of Theorem 3.4. To simplify the notation, we introduce a intermediate operator defined as

$$(3.31) \qquad L_{t,h} u(\mathbf{x}) = \frac{1}{t} \sum_{\mathbf{x}_j \in P} R_t(\mathbf{x}, \mathbf{x}_j)(u(\mathbf{x}) - u(\mathbf{x}_j)) p(\mathbf{x}_j) V_j.$$

Let $u_{t,h} = I_{\mathbf{f}}(\mathbf{u})$ with $\mathbf{u}$ satisfying (3.2) and $I_{\mathbf{f}}$ given in (3.3). One can verify that the

following equation is satisfied:

$$(3.32) \qquad -L_{t,h}u_{t,h}(\mathbf{x}) = \sum_{\mathbf{x}_j \in P} \bar{R}_t(\mathbf{x}, \mathbf{x}_j) f(\mathbf{x}_j)/p(\mathbf{x}_j) V_j.$$

In the proof, we need *a prior* estimate of **u** which is given as follows.

THEOREM 3.8. *Suppose* $\mathbf{u} = (u_1, \ldots, u_{|P|})$ *with* $\sum_{i=1}^{|P|} u_i p_i V_i = 0$ *solves the problem* (3.2) *and* $\mathbf{f} = (f(\mathbf{x}_1), \ldots, f(\mathbf{x}_{|P|}))^t$ *for* $f \in C(\mathcal{M})$. *Then there exists a constant* $C > 0$ *such that*

$$\left(\sum_{i=1}^{|P|} u_i^2 p_i V_i \right)^{1/2} \le C \|f\|_\infty,$$

provided t *and* $\frac{h(P, \mathbf{V}, \mathcal{M})}{\sqrt{t}}$ *are small enough.*

This theorem is an easy corollary of following theorem.

THEOREM 3.9. *If the manifold* $\mathcal{M}$ *is* C^∞, *there exist constants* $C > 0$, $C_0 > 0$ *independent on* t *so that for any* $\mathbf{u} = (u_1, \ldots, u_{|P|})^t \in \mathbb{R}^d$ *with* $\sum_{i=1}^{|P|} u_i p_i V_i = 0$ *and for any sufficient small* t *and* $\frac{h(P, \mathbf{V}, \mathcal{M})}{\sqrt{t}}$

$$\sum_{i,j=1}^{|P|} R_t(\mathbf{x}_i, \mathbf{x}_j)(u_i - u_j)^2 p_i p_j V_i V_j \ge C \left(1 - \frac{C_0 h(P, \mathbf{V}, \mathcal{M})}{\sqrt{t}} \right) \sum_{i=1}^{|P|} u_i^2 p_i V_i.$$

The proof of this theorem is given in the supplementary material which is a small modification of the proof of Theorem 9.1 in [33].

We are now ready to prove Theorem 3.4.

Proof. To simplify the notation, we denote $h = h(P, \mathbf{V}, \mathcal{M})$ and $n = |P|$ and denote

$$(3.33) \quad u_{t,h}(\mathbf{x}) = \frac{1}{w_{t,h}(\mathbf{x})} \left(\sum_{\mathbf{x}_j \in P} R_t(\mathbf{x}, \mathbf{x}_j) u_j p_j V_j - t \sum_{\mathbf{x}_j \in P} \bar{R}_t(\mathbf{x}, \mathbf{x}_j) f_j V_j/p_j \right),$$

where $\mathbf{u} = (u_1, \ldots, u_n)^t$ with $\sum_{i=1}^n u_i p_i V_i = 0$ solves the problem (3.2), $f_j = f(\mathbf{x}_j)$ and $w_{t,h}(\mathbf{x}) = \sum_{\mathbf{x}_j \in P} R_t(\mathbf{x}, \mathbf{x}_j) p_j V_j$. For convenience, we set

$$(3.34) \qquad a_{t,h}(\mathbf{x}) = \frac{1}{w_{t,h}(\mathbf{x})} \sum_{\mathbf{x}_j \in P} R_t(\mathbf{x}, \mathbf{x}_j) u_j p_j V_j,$$

$$(3.35) \qquad c_{t,h}(\mathbf{x}) = -\frac{t}{w_{t,h}(\mathbf{x})} \sum_{\mathbf{x}_j \in P} \bar{R}_t(\mathbf{x}, \mathbf{x}_j) f(\mathbf{x}_j) V_j/p_j,$$

and thus $u_{t,h} = a_{t,h} + c_{t,h}$.

First we upper bound $\|L_t(u_{t,h}) - L_{t,h}(u_{t,h})\|_{L^2(\mathcal{M})}$. For $c_{t,h}$, we have

$$|(L_t c_{t,h} - L_{t,h} c_{t,h})(\mathbf{x})|$$

$$= \frac{1}{t}\left| \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y})(c_{t,h}(\mathbf{x}) - c_{t,h}(\mathbf{y}))p(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \right.$$

$$\left. - \sum_{\mathbf{x}_j \in P} R_t(\mathbf{x}, \mathbf{x}_j)(c_{t,h}(\mathbf{x}) - c_{t,h}(\mathbf{x}_j))p_j V_j \right|$$

$$\leq \frac{1}{t}|c_{t,h}(\mathbf{x})| \left| \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y})p(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} - \sum_{\mathbf{x}_j \in P} R_t(\mathbf{x}, \mathbf{x}_j)p_j V_j \right|$$

$$+ \frac{1}{t}\left| \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y})c_{t,h}(\mathbf{y})p(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} - \sum_{\mathbf{x}_j \in P} R_t(\mathbf{x}, \mathbf{x}_j)c_{t,h}(\mathbf{x}_j)p_j V_j \right|$$

$$\leq \frac{Ch}{t^{3/2}}|c_{t,h}(\mathbf{x})| + \frac{Ch}{t^{3/2}}\|c_{t,h}\|_{C^1(\mathcal{M})}$$

$$\leq \frac{Ch}{t^{3/2}}t\|f\|_\infty + \frac{Ch}{t^{3/2}}(t\|f\|_\infty + t^{1/2}\|f\|_\infty) \leq \frac{Ch}{t}\|f\|_\infty.$$

For $a_{t,h}$, we have

$$(3.36) \quad \int_{\mathcal{M}} (a_{t,h}(\mathbf{x}))^2 \left| \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y})p(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} - \sum_{\mathbf{x}_j \in P} R_t(\mathbf{x}, \mathbf{x}_j)p_j V_j \right|^2 \mathrm{d}\mu_{\mathbf{x}}$$

$$\leq \frac{Ch^2}{t} \int_{\mathcal{M}} (a_{t,h}(\mathbf{x}))^2 \, \mathrm{d}\mu_{\mathbf{x}}$$

$$\leq \frac{Ch^2}{t} \int_{\mathcal{M}} \left(\frac{1}{w_{t,h}(\mathbf{x})} \sum_{\mathbf{x}_j \in P} R_t(\mathbf{x}, \mathbf{x}_j)u_j p_j V_j \right)^2 \mathrm{d}\mu_{\mathbf{x}}$$

$$\leq \frac{Ch^2}{t} \int_{\mathcal{M}} \left(\sum_{\mathbf{x}_j \in P} R_t(\mathbf{x}, \mathbf{x}_j)u_j^2 p_j V_j \right) \left(\sum_{\mathbf{x}_j \in P} R_t(\mathbf{x}, \mathbf{x}_j)p_j V_j \right) \mathrm{d}\mu_{\mathbf{x}}$$

$$\leq \frac{Ch^2}{t} \left(\sum_{j=1}^n u_j^2 p_j V_j \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{x}_j)\mathrm{d}\mu_{\mathbf{x}} \right) \leq \frac{Ch^2}{t} \sum_{j=1}^n u_j^2 p_j V_j.$$

Let

$$A = C_t \int_{\mathcal{M}} \frac{1}{w_{t,h}(\mathbf{y})} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t} \right) R\left(\frac{|\mathbf{x}_i - \mathbf{y}|^2}{4t} \right) p(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}$$

$$- C_t \sum_{\mathbf{x}_j \in P} \frac{1}{w_{t,h}(\mathbf{x}_j)} R\left(\frac{|\mathbf{x} - \mathbf{x}_j|^2}{4t} \right) R\left(\frac{|\mathbf{x}_i - \mathbf{x}_j|^2}{4t} \right) p_j V_j.$$

We have $|A| < \frac{Ch}{t^{1/2}}$ for some constant C independent of t. In addition, notice that only when $|\mathbf{x} - \mathbf{x}_i|^2 \leq 16t$ is $A \neq 0$, which implies

$$|A| \leq \frac{1}{\delta_0}|A|R\left(\frac{|\mathbf{x} - \mathbf{x}_i|^2}{32t} \right).$$

Then we have

(3.37)
$$\int_{\mathcal{M}} \left| \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) a_{t,h}(\mathbf{y}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} - \sum_{\mathbf{x}_j \in P} R_t(\mathbf{x}, \mathbf{x}_j) a_{t,h}(\mathbf{x}_j) p_j V_j \right|^2 \mathrm{d}\mu_{\mathbf{x}}$$

$$= \int_{\mathcal{M}} \left(\sum_{i=1}^n C_t u_i p_i V_i A \right)^2 \mathrm{d}\mu_{\mathbf{x}} \le \frac{Ch^2}{t} \int_{\mathcal{M}} \left(\sum_{i=1}^n C_t |u_i| p_i V_i R\left(\frac{|\mathbf{x} - \mathbf{x}_i|^2}{32t} \right) \right)^2 \mathrm{d}\mu_{\mathbf{x}}$$

$$\le \frac{Ch^2}{t} \int_{\mathcal{M}} \left(\sum_{i=1}^n C_t R\left(\frac{|\mathbf{x} - \mathbf{x}_i|^2}{32t} \right) u_i^2 p_i V_i \right) \left(\sum_{\mathbf{x}_i \in P} C_t R\left(\frac{|\mathbf{x} - \mathbf{x}_i|^2}{32t} \right) p_i V_i \right) \mathrm{d}\mu_{\mathbf{x}}$$

$$\le \frac{Ch^2}{t} \sum_{i=1}^n \left(\int_{\mathcal{M}} C_t R\left(\frac{|\mathbf{x} - \mathbf{x}_i|^2}{32t} \right) \mathrm{d}\mu_{\mathbf{x}} \left(u_i^2 p_i V_i \right) \right) \le \frac{Ch^2}{t} \left(\sum_{i=1}^n u_i^2 p_i V_i \right).$$

Combining (3.36), (3.37), and Lemma 3.8,

$$\|L_t a_{t,h} - L_{t,h} a_{t,h}\|_{L^2(\mathcal{M})}$$

$$= \left(\int_{\mathcal{M}} |(L_t(a_{t,h}) - L_{t,h}(a_{t,h}))(\mathbf{x})|^2 \mathrm{d}\mu_{\mathbf{x}} \right)^{1/2}$$

$$\le \frac{1}{t} \left(\int_{\mathcal{M}} (a_{t,h}(\mathbf{x}))^2 \left| \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} - \sum_{\mathbf{x}_j \in P} R_t(\mathbf{x}, \mathbf{x}_j) p_j V_j \right|^2 \mathrm{d}\mu_{\mathbf{x}} \right)^{1/2}$$

$$+ \frac{1}{t} \left(\int_{\mathcal{M}} \left| \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) a_{t,h}(\mathbf{y}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} - \sum_{\mathbf{x}_j \in P} R_t(\mathbf{x}, \mathbf{x}_j) a_{t,h}(\mathbf{x}_j) p_j V_j \right|^2 \mathrm{d}\mu_{\mathbf{x}} \right)^{1/2}$$

$$\le \frac{Ch}{t^{3/2}} \left(\sum_{i=1}^n u_i^2 p_i V_i \right)^{1/2} \le \frac{Ch}{t^{3/2}} \|f\|_\infty.$$

Assembling the parts together, we have the following upper bound:

(3.38)
$$\|L_t u_{t,h} - L_{t,h} u_{t,h}\|_{L^2(\mathcal{M})}$$
$$\le \|L_t a_{t,h} - L_{t,h} a_{t,h}\|_{L^2(\mathcal{M})} + \|L_t c_{t,h} - L_{t,h} c_{t,h}\|_{L^2(\mathcal{M})}$$
$$\le \frac{Ch}{t^{3/2}} \|f\|_\infty + \frac{Ch}{t} \|f\|_\infty \le \frac{Ch}{t^{3/2}} \|f\|_\infty.$$

At the same time, since u_t, respectively, $u_{t,h}$, solves (3.5), respectively, (3.32), we have

(3.39) $\|L_t(u_t) - L_{t,h}(u_{t,h})\|_{L^2(\mathcal{M})}$

$$= \left(\int_{\mathcal{M}} ((L_t u_t - L_{t,h} u_{t,h})(\mathbf{x}))^2 \mathrm{d}\mu_{\mathbf{x}} \right)^{1/2}$$

$$= \left(\int_{\mathcal{M}} \left(\int_{\mathcal{M}} \bar{R}_t(\mathbf{x}, \mathbf{y}) f(\mathbf{y})/p(\mathbf{y}) - \sum_{\mathbf{x}_j \in P} \bar{R}_t(\mathbf{x}, \mathbf{x}_j) f(\mathbf{x}_j) V_j/p_j \right)^2 \mathrm{d}\mu_{\mathbf{x}} \right)^{1/2}$$

$$\le \frac{Ch}{t^{1/2}} \|f\|_{C^1(\mathcal{M})}.$$

The complete L^2 estimate follows from (3.38) and (3.39).

The estimate of the gradient, $\|\nabla(L_t(u_t) - L_{t,h}(u_{t,h}))\|_{L^2(\mathcal{M})}$, can be obtained similarly. □

3.5. Proof of Theorem 3.3. In order to prove Theorem 3.3, we need two theorems, 3.10 and 3.11. The proof of these two theorems can be obtained by making a minor revision of the proof of Theorems 4.4 and 4.5 in [33]; the details of the proof are put in the supplementary material.

THEOREM 3.10. *For any function $u \in L^2(\mathcal{M})$, there exists a constant $C > 0$ independent on t and u, such that*

$$\int_{\mathcal{M}} \int_{\mathcal{M}} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t}\right) (u(\mathbf{x}) - u(\mathbf{y}))^2 p(\mathbf{x}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\mu_{\mathbf{y}} \geq C \int_{\mathcal{M}} |\nabla v(\mathbf{x})|^2 p(\mathbf{x}) \mathrm{d}\mu_{\mathbf{x}},$$

where

$$v(\mathbf{x}) = \frac{C_t}{w_t(\mathbf{x})} \int_{\mathcal{M}} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t}\right) u(\mathbf{y}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}},$$

and $w_t(\mathbf{x}) = C_t \int_{\mathcal{M}} R(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}}$.

THEOREM 3.11. *Assume both $\mathcal{M}$ and $\partial\mathcal{M}$ are C^{∞}. There exists a constant $C > 0$ independent on t so that for any function $u \in L_2(\mathcal{M})$ with $\int_{\mathcal{M}} u(\mathbf{x}) p(\mathbf{x}) \mathrm{d}\mu_{\mathbf{x}} = 0$ and for any sufficient small t*

$$\int_{\mathcal{M}} \int_{\mathcal{M}} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t}\right) (u(\mathbf{x}) - u(\mathbf{y}))^2 p(\mathbf{x}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\mu_{\mathbf{y}} \geq C \|u\|_{L_2(\mathcal{M})}^2.$$

Using the above two theorems, Theorem 3.3 becomes an easy corollary.

Proof of Theorem 3.3. Using Theorem 3.11, we have

$$(3.40) \qquad \|u\|_{L^2(\mathcal{M})}^2 \leq C \int_{\mathcal{M}} u(\mathbf{x}) r(\mathbf{x}) p(\mathbf{x}) \mathrm{d}\mu_{\mathbf{x}} \leq C \|u\|_{L^2(\mathcal{M})} \|r\|_{L^2(\mathcal{M})}.$$

This inequality (3.40) implies that

$$\|u\|_{L^2(\mathcal{M})} \leq C \|r\|_{L^2(\mathcal{M})}.$$

Now we turn to estimate $\|\nabla u\|_{L^2(\mathcal{M})}$. Notice that we have the following expression for u:

$$u(\mathbf{x}) = v(\mathbf{x}) + \frac{t}{w_t(\mathbf{x})} r(\mathbf{x}),$$

where

$$v(\mathbf{x}) = \frac{1}{w_t(\mathbf{x})} \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) u(\mathbf{y}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}}, \quad w_t(\mathbf{x}) = \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}}.$$

By Theorem 3.10, we have

$$\|\nabla u\|_{L^2(\mathcal{M})}^2 \leq 2\|\nabla v\|_{L^2(\mathcal{M})}^2 + 2t^2 \left\|\nabla\left(\frac{r(\mathbf{x}) - \bar{r}}{w_t(\mathbf{x})}\right)\right\|_{L^2(\mathcal{M})}^2$$

$$\leq C \int_{\mathcal{M}} u(\mathbf{x}) L_t u(\mathbf{x}) p(\mathbf{x}) \mathrm{d}\mu_{\mathbf{x}} + Ct\|r\|_{L^2(\mathcal{M})}^2 + Ct^2\|\nabla r\|_{L^2(\mathcal{M})}^2$$

$$\leq C\|u\|_{L^2(\mathcal{M})} \|r\|_{L^2(\mathcal{M})} + Ct\|r\|_{L^2(\mathcal{M})}^2 + Ct^2\|\nabla r\|_{L^2(\mathcal{M})}^2$$

$$\leq C\|r\|_{L^2(\mathcal{M})}^2 + Ct^2\|\nabla r\|_{L^2(\mathcal{M})}^2$$

$$\leq C \left(\|r\|_{L^2(\mathcal{M})} + t\|\nabla r\|_{L^2(\mathcal{M})}\right)^2.$$

The proof is completed. □

3.6. Proof of Theorem 3.6.

Proof. First, we denote

$$r(\mathbf{x}) = \int_{\partial \mathcal{M}} \mathbf{b}(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \bar{R}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}},$$

$$\bar{r} = \frac{1}{|\mathcal{M}|_p} \int_{\mathcal{M}} \left(\int_{\partial \mathcal{M}} \mathbf{b}(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \bar{R}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}} \right) p(\mathbf{x}) \mathrm{d}\mathbf{x},$$

where $|\mathcal{M}|_p = \int_{\mathcal{M}} p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}}$.

The key point of the proof is to show that

$$(3.41) \qquad \left| \int_{\mathcal{M}} u(\mathbf{x}) \left(r(\mathbf{x}) - \bar{r} \right) p(\mathbf{x}) \mathrm{d}\mu_{\mathbf{x}} \right| \leq C\sqrt{t} \, \|\mathbf{b}\|_{H^1(\mathcal{M})} \|u\|_{H^1(\mathcal{M})}.$$

First, notice that

$$|\bar{r}| \leq C\sqrt{t} \, \|\mathbf{b}\|_{L^2(\partial \mathcal{M})} \leq C\sqrt{t} \, \|\mathbf{b}\|_{H^1(\mathcal{M})}.$$

Then it is sufficient to show that
(3.42)
$$\left| \int_{\mathcal{M}} u(\mathbf{x}) \left(\int_{\partial \mathcal{M}} \mathbf{b}(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \bar{R}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}} \right) p(\mathbf{x}) \mathrm{d}\mu_{\mathbf{x}} \right| \leq C\sqrt{t} \, \|\mathbf{b}\|_{H^1(\mathcal{M})} \|u\|_{H^1(\mathcal{M})}.$$

Direct calculation gives that

$$|2t\nabla \bar{\bar{R}}_t(\mathbf{x}, \mathbf{y}) - (\mathbf{x} - \mathbf{y}) \bar{R}_t(\mathbf{x}, \mathbf{y})| \leq C|\mathbf{x} - \mathbf{y}|^2 \bar{R}_t(\mathbf{x}, \mathbf{y}),$$

where $\bar{\bar{R}}_t(\mathbf{x}, \mathbf{y}) = C_t \bar{\bar{R}}(\frac{\|\mathbf{x} - \mathbf{y}\|^2}{4t})$ and $\bar{\bar{R}}(r) = \int_r^\infty \bar{R}(s) \mathrm{d}s$. This implies that

$$(3.43) \qquad \left| \int_{\mathcal{M}} u(\mathbf{x}) p(\mathbf{x}) \int_{\partial \mathcal{M}} \mathbf{b}(\mathbf{y}) \left((\mathbf{x} - \mathbf{y}) \bar{R}_t(\mathbf{x}, \mathbf{y}) + 2t\nabla \bar{\bar{R}}_t(\mathbf{x}, \mathbf{y}) \right) p(\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{x}} \right|$$

$$\leq C \int_{\mathcal{M}} |u(\mathbf{x}) p(\mathbf{x})| \int_{\partial \mathcal{M}} |\mathbf{b}(\mathbf{y})| |\mathbf{x} - \mathbf{y}|^2 \bar{R}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{x}}$$

$$\leq Ct\|\mathbf{b}\|_{L^2(\partial \mathcal{M})} \left(\int_{\partial \mathcal{M}} \left(\int_{\mathcal{M}} \bar{R}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{x}) \mathrm{d}\mu_{\mathbf{x}} \right) \right.$$

$$\left. \left(\int_{\mathcal{M}} |u(\mathbf{x})|^2 \bar{R}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{x}) \mathrm{d}\mu_{\mathbf{x}} \right) p(\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}} \right)^{1/2}$$

$$\leq Ct\|\mathbf{b}\|_{H^1(\mathcal{M})} \left(\int_{\mathcal{M}} |u(\mathbf{x})|^2 p(\mathbf{x}) \left(\int_{\partial \mathcal{M}} \bar{R}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}} \right) \mathrm{d}\mu_{\mathbf{x}} \right)^{1/2}$$

$$\leq Ct^{3/4}\|\mathbf{b}\|_{H^1(\mathcal{M})} \|u\|_{L^2(\mathcal{M})}.$$

On the other hand, using the Gauss integral formula, we have

$$(3.44) \qquad \int_{\mathcal{M}} u(\mathbf{x}) p(\mathbf{x}) \int_{\partial \mathcal{M}} \mathbf{b}(\mathbf{y}) \cdot \nabla \bar{\bar{R}}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{x}}$$

$$= \int_{\partial \mathcal{M}} \int_{\mathcal{M}} u(\mathbf{x}) p(\mathbf{x}) T_{\mathbf{x}}(\mathbf{b}(\mathbf{y})) \cdot \nabla \bar{\bar{R}}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\tau_{\mathbf{y}}$$

$$= \int_{\partial \mathcal{M}} \int_{\partial \mathcal{M}} \mathbf{n}(\mathbf{x}) \cdot T_{\mathbf{x}}(\mathbf{b}(\mathbf{y})) u(\mathbf{x}) \bar{\bar{R}}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{x}) p(\mathbf{y}) \mathrm{d}\tau_{\mathbf{x}} \mathrm{d}\tau_{\mathbf{y}}$$

$$- \int_{\partial \mathcal{M}} \int_{\mathcal{M}} \mathrm{div}_{\mathbf{x}}[u(\mathbf{x}) p(\mathbf{x}) T_{\mathbf{x}}(\mathbf{b}(\mathbf{y}))] \bar{\bar{R}}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\tau_{\mathbf{y}}.$$

Here $T_{\mathbf{x}}$ is the projection operator to the tangent space on $\mathbf{x}$. To get the first equality, we use the fact that $\nabla \bar{\bar{R}}_t(\mathbf{x}, \mathbf{y})$ belongs to the tangent space on $\mathbf{x}$, such that $\mathbf{b}(\mathbf{y}) \cdot \nabla \bar{\bar{R}}_t(\mathbf{x}, \mathbf{y}) = T_{\mathbf{x}}(\mathbf{b}(\mathbf{y})) \cdot \nabla \bar{\bar{R}}_t(\mathbf{x}, \mathbf{y})$ and $\mathbf{n}(\mathbf{x}) \cdot T_{\mathbf{x}}(\mathbf{b}(\mathbf{y})) = \mathbf{n}(\mathbf{x}) \cdot \mathbf{b}(\mathbf{y})$, where $\mathbf{n}(\mathbf{x})$ is the out normal of $\partial \mathcal{M}$ at $\mathbf{x} \in \partial \mathcal{M}$.

For the first term, we have

$$(3.45) \quad \left| \int_{\partial \mathcal{M}} \int_{\partial \mathcal{M}} \mathbf{n}(\mathbf{x}) \cdot T_{\mathbf{x}}(\mathbf{b}(\mathbf{y})) u(\mathbf{x}) \bar{\bar{R}}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{x}) p(\mathbf{y}) \mathrm{d}\tau_{\mathbf{x}} \mathrm{d}\tau_{\mathbf{y}} \right|$$

$$= \left| \int_{\partial \mathcal{M}} \int_{\partial \mathcal{M}} \mathbf{n}(\mathbf{x}) \cdot \mathbf{b}(\mathbf{y}) u(\mathbf{x}) \bar{\bar{R}}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{x}) p(\mathbf{y}) \mathrm{d}\tau_{\mathbf{x}} \mathrm{d}\tau_{\mathbf{y}} \right|$$

$$\leq C \|\mathbf{b}\|_{L^2(\partial \mathcal{M})} \left(\int_{\partial \mathcal{M}} \left(\int_{\partial \mathcal{M}} |u(\mathbf{x})| \bar{\bar{R}}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{x}) \mathrm{d}\tau_{\mathbf{x}} \right)^2 p(\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}} \right)^{1/2}$$

$$\leq C \|\mathbf{b}\|_{H^1(\mathcal{M})} \left(\int_{\partial \mathcal{M}} \left(\int_{\partial \mathcal{M}} \bar{\bar{R}}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{x}) \mathrm{d}\tau_{\mathbf{x}} \right) \right.$$

$$\left. \left(\int_{\partial \mathcal{M}} |u(\mathbf{x})|^2 \bar{\bar{R}}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{x}) \mathrm{d}\tau_{\mathbf{x}} \right) p(\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}} \right)^{1/2}$$

$$\leq C t^{-1/2} \|\mathbf{b}\|_{H^1(\mathcal{M})} \|u\|_{L^2(\partial \mathcal{M})} \leq C t^{-1/2} \|\mathbf{b}\|_{H^1(\mathcal{M})} \|u\|_{H^1(\mathcal{M})}.$$

We can also bound the second term on the right-hand side of (3.44). By using the assumption that $\mathcal{M} \in C^\infty$, we have

$$|\mathrm{div}_{\mathbf{x}}[u(\mathbf{x}) p(\mathbf{x}) T_{\mathbf{x}}(\mathbf{b}(\mathbf{y}))]|$$
$$\leq |\nabla u(\mathbf{x})| |T_{\mathbf{x}}(\mathbf{b}(\mathbf{y}))| |p(\mathbf{x})| + |u(\mathbf{x})| |\mathrm{div}_{\mathbf{x}}[T_{\mathbf{x}}(\mathbf{b}(\mathbf{y}))]| |p(\mathbf{x})| + |\nabla p(\mathbf{x})| |u(\mathbf{x}) T_{\mathbf{x}}(\mathbf{b}(\mathbf{y}))|$$
$$\leq C(|\nabla u(\mathbf{x})| + |u(\mathbf{x})|) |\mathbf{b}(\mathbf{y})|,$$

where the constant C depends on the curvature of the manifold $\mathcal{M}$.

Then, we have

$$(3.46) \quad \left| \int_{\partial \mathcal{M}} \int_{\mathcal{M}} \mathrm{div}_{\mathbf{x}}[u(\mathbf{x}) T_{\mathbf{x}}(\mathbf{b}(\mathbf{y}))] \bar{\bar{R}}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{x}) p(\mathbf{y}) \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\tau_{\mathbf{y}} \right|$$

$$\leq C \int_{\partial \mathcal{M}} \mathbf{b}(\mathbf{y}) p(\mathbf{y}) \int_{\mathcal{M}} (|\nabla u(\mathbf{x})| + |u(\mathbf{x})|) \bar{\bar{R}}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{x}) \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\tau_{\mathbf{y}}$$

$$\leq C \|\mathbf{b}\|_{L^2(\partial \mathcal{M})} \left(\int_{\mathcal{M}} (|\nabla u(\mathbf{x})|^2 + |u(\mathbf{x})|^2) p(\mathbf{x}) \right.$$

$$\left. \left(\int_{\partial \mathcal{M}} \bar{\bar{R}}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}} \right) \mathrm{d}\mu_{\mathbf{x}} \right)^{1/2}$$

$$\leq C t^{-1/4} \|\mathbf{b}\|_{H^1(\mathcal{M})} \|u\|_{H^1(\mathcal{M})}.$$

Then, the inequality (3.42) is obtained from (3.43), (3.44), (3.45), and (3.46). Now, using Theorem 3.11, we have

$$(3.47) \quad \|u\|_{L^2(\mathcal{M})}^2 \leq C \int_{\mathcal{M}} u(\mathbf{x}) L_t u(\mathbf{x}) p(\mathbf{x}) \mathrm{d}\mu_{\mathbf{x}} \leq C \sqrt{t} \|\mathbf{b}\|_{H^1(\mathcal{M})} \|u\|_{H^1(\mathcal{M})}.$$

Note $r(\mathbf{x}) = \int_{\partial \mathcal{M}} (\mathbf{x} - \mathbf{y}) \cdot \mathbf{b}(\mathbf{y}) \bar{R}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}}$. Direct calculation gives us that

$$\|r(\mathbf{x})\|_{L^2(\mathcal{M})} \leq C t^{1/4} \|\mathbf{b}\|_{H^1(\mathcal{M})} \text{ and}$$
$$\|\nabla r(\mathbf{x})\|_{L^2(\mathcal{M})} \leq C t^{-1/4} \|\mathbf{b}\|_{H^1(\mathcal{M})}.$$

The integral equation $L_t u = r - \bar{r}$ gives that

$$u(\mathbf{x}) = v(\mathbf{x}) + \frac{t}{w_t(\mathbf{x})}\left(r(\mathbf{x}) - \bar{r}\right),$$

where

$$v(\mathbf{x}) = \frac{1}{w_t(\mathbf{x})}\int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y})u(\mathbf{y})p(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}, \quad w_t(\mathbf{x}) = \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y})p(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}.$$

By Theorem 3.10, we have

$$\begin{aligned}
(3.48) \quad &\|\nabla u\|_{L^2(\mathcal{M})}^2 \\
&\leq 2\|\nabla v\|_{L^2(\mathcal{M})}^2 + 2t^2 \left\|\nabla\left(\frac{r(\mathbf{x}) - \bar{r}}{w_t(\mathbf{x})}\right)\right\|_{L^2(\mathcal{M})}^2 \\
&\leq C\int_{\mathcal{M}} u(\mathbf{x})L_t u(\mathbf{x})p(\mathbf{x})\mathrm{d}\mu_{\mathbf{x}} + Ct\|r\|_{L^2(\mathcal{M})}^2 + Ct^2\|\nabla r\|_{L^2(\mathcal{M})}^2 \\
&\leq C\sqrt{t}\,\|\mathbf{b}\|_{H^1(\mathcal{M})}\|u\|_{H^1(\mathcal{M})} + Ct\|r\|_{L^2(\mathcal{M})}^2 + Ct^2\|\nabla r\|_{L^2(\mathcal{M})}^2 \\
&\leq C\|\mathbf{b}\|_{H^1(\mathcal{M})}\left(\sqrt{t}\|u\|_{H^1(\mathcal{M})} + Ct^{3/2}\right).
\end{aligned}$$

Using (3.47) and (3.48), we have

$$\|u\|_{H^1(\mathcal{M})}^2 \leq C\|\mathbf{b}\|_{H^1(\mathcal{M})}\left(\sqrt{t}\|u\|_{H^1(\mathcal{M})} + Ct^{3/2}\right),$$

which proves the theorem. $\qquad\square$

4. Numerical experiments. In this section, we show several numerical examples to demonstrate the performance of the PIM for isotropic elliptic equations. This section is separated to two parts. In the first part, on some simple 2D surfaces, the convergence of the PIM is verified. In the second part, we consider a nonlocal total variation minimization problem, in which some isotropic elliptic equations are solved on point cloud in high dimensional space.

4.1. Examples on 2D surfaces. In this subsection, we consider the isotropic elliptic equation on 2D surfaces

$$(4.1) \qquad\qquad -\mathrm{div}(p^2(\mathbf{x})\nabla u(\mathbf{x})) = f(\mathbf{x}), \quad \mathbf{x} \in \mathcal{M},$$

with Neumann and Dirichlet boundary conditions,

$$\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = b(\mathbf{x}) \quad \text{or} \quad u(\mathbf{x}) = b(\mathbf{x}), \quad \mathbf{x} \in \partial\mathcal{M}$$

To estimate the volume weight vector $\mathbf{V}$ from the point sets P, a local mesh around each sample point is constructed, from which the weight of that point is computed. For details to estimate the volume weight, we refer to [26]. The kernel function is chosen to be Gaussian function,

$$R_t(\mathbf{x}, \mathbf{y}) = \frac{1}{(4\pi t)^{k/2}}\exp\left(-\frac{\|\mathbf{x} - \mathbf{y}\|^2}{4t}\right).$$

The parameter t is set as $t = \left(\frac{1}{|P|}\sum_{i=1}^{|P|}\rho(\mathbf{x}_i)\right)^2$, where $\rho(\mathbf{x}_i)$ is the radius of 10 nearest neighbors of $\mathbf{x}_i$.
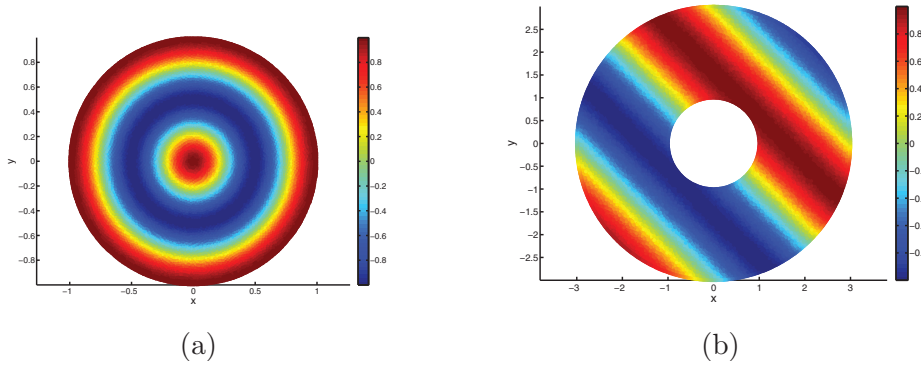
(a)   (b)

FIG. 1. *Ground truth:* (a) $u_{gt} = \cos(2\pi r)$ *in unit disk;* (b) $u_{gt} = \sin(x+y)$ *in the annulus.*

TABLE 1
l_2 error for $u_{gt} = \cos(2\pi r)$ in the unit disk and $\sin(x+y)$ in the annulus.

$|P|$	684	2610	10191	40296
Disk	0.364597	0.214960	0.111961	0.056028
Annulus	0.036760	0.012227	0.005557	0.003542

Example 1. In the first example, the manifold $\mathcal{M}$ is an unit disk and an annulus in $\mathbb{R}^2$. The inner radius of the annulus is 1, and the outer radius is 3. The exact solution is set to be $u_{gt}(\mathbf{x}) = \cos(2\pi\|\mathbf{x}\|)$ in unit disk and $u_{gt} = \sin(x+y)$ in the annulus; see Figure 1. The coefficient of the equation in (4.1) is

$$(4.2) \qquad\qquad p = 1 + \frac{1}{4}\|\mathbf{x}\|^2,$$

both in the unit disk and annulus. The Neumann boundary condition is enforced in unit disk and we consider the Dirichlet boundary condition in the annulus.

Table 1 lists the l_2 error of the PIM as the number of points grows. This result clearly shows the convergence of the PIM. The convergence rate in l_2 error is approximately $1/\sqrt{|P|}$.

The eigenvalue problem with homogeneous Neumann boundary condition is also solved in the annulus.

$$-\mathrm{div}(p^2(\mathbf{x})\nabla u(\mathbf{x})) = \lambda u(\mathbf{x}), \quad \mathbf{x} \in \mathcal{M},$$
$$\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = 0, \qquad \mathbf{x} \in \partial\mathcal{M},$$

and the coefficient p is given in (4.2).

The first 20 eigenvalues are plotted in Figure 2. The eigenvalues given by finite element method in the finest mesh is used as the true solution. Our result shows that the eigenvalue computed in the PIM also converge.

Example 2. Now, we solve (4.1) with Neumann condition and Dirichlet condition on a curved surface in $\mathbb{R}^3$. Let $\mathcal{M}$ be a cap on the unit sphere, whose height is $1/2$ and the cap angle is $\pi/3$, as shown in Figure 3. The coefficient of the equation is also given in (4.2).

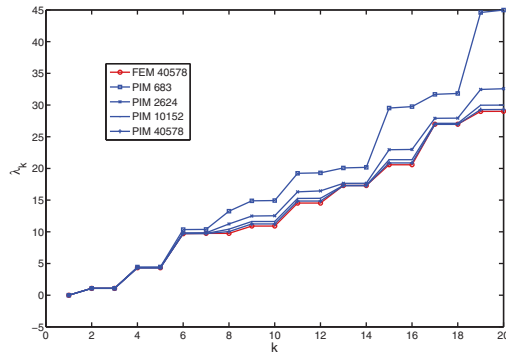We set the ground truth to be $u_{gt} = x + y + z$, where (x, y, z) is the coordinate in $\mathbb{R}^3$.

FIG. 2. *First* 20 *eigenvalues in the annulus with Neumann boundary condition with different point cloud.*
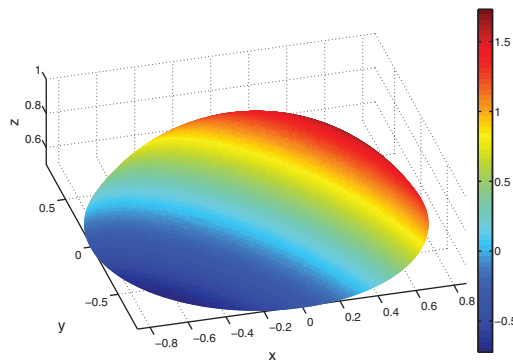

FIG. 3. *Ground truth:* $u_{gt} = x + y + z$ *on the cap.*

TABLE 2
l_2 *error for* $u_{gt} = x + y + z$ *on cap.*

$|P|$	1199	4689	18540	73757
Neumann	0.036779	0.015355	0.007479	0.003189
Dirichlet	0.007238	0.001921	0.001278	0.000750

The l_2 errors of the PIM are listed in Table 2. The convergence rate for both boundary value problems is 1.

The first 20 eigenvalues are also computed for the homogeneous Neumann condition as shown in Figure 4. As the number of points increases, the eigenvalues given by PIM converge to those computed by FEM, which suggests the convergence of the PIM.

Example 3. In this example, we consider a more complex surface, a human face called "Alex." The surface is sampled by 10,597 points (Figure 5), and the analytic form of the surface is not known. The coefficient of the equation in (4.1) is

$$p^2 = \frac{1}{\sin(r/10)/2 + 1},$$

where $r = \sqrt{x^2 + y^2 + z^2}$. In this example, we solve the eigenvalue problem of the isotropic elliptic operator. Several eigenfunctions computed by the PIM are shown in Figure 6.
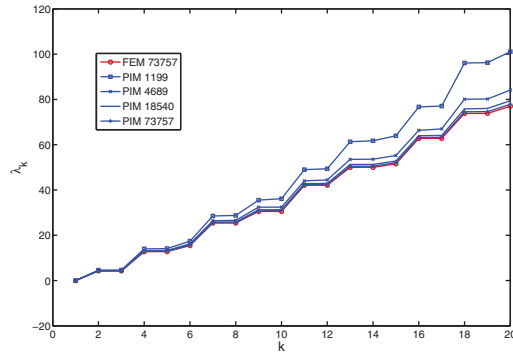
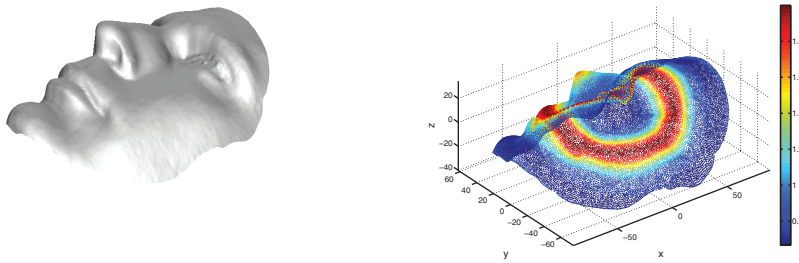FIG. 4. *First* 20 *eigenvalues on the cap with Neumann boundary condition.*

FIG. 5. (a) *Face of Alex;* (b) *coefficient: restriction of* p^2 *on* $\mathcal{M}$.
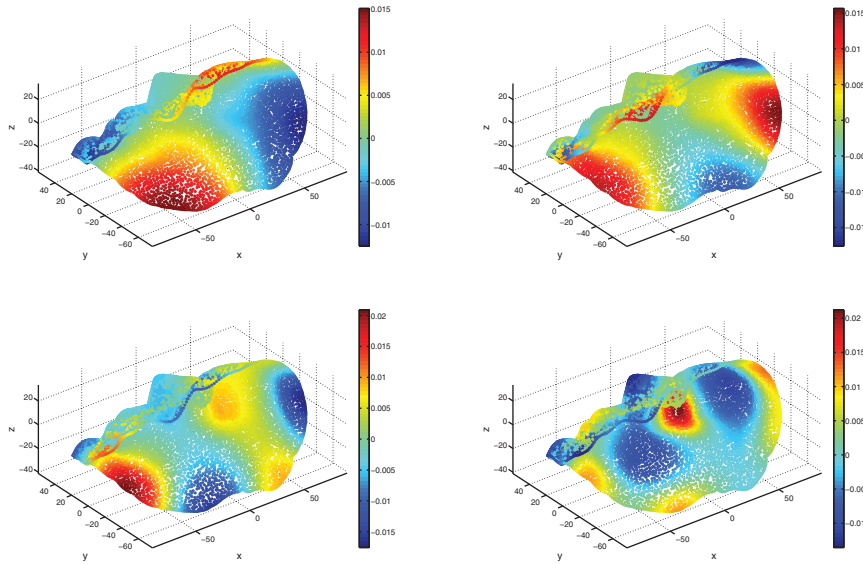

FIG. 6. *Eigenfunctions on Alex with homogenous Neumann boundary condition.*

From the examples in 2D surfaces, we see that PIM solves isotropic elliptic equations with a Neumann and Dirichlet boundary very well. Moreover, the convergence rate is higher than that obtained in the convergence analysis. The PIM is applicable to point cloud in high dimensional space, not only on the 2D surfaces. Next, we will show a high dimensional example.
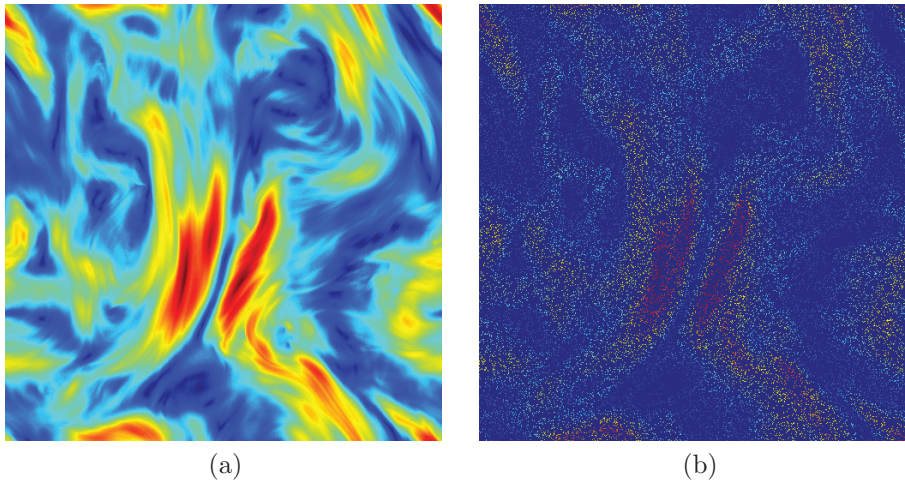
(a)   (b)

FIG. 7. (a): *original data;* (b): *10% subsampled data.*

4.2. Nonlocal total variation extension. In this example, we consider an L_1 extension on a point cloud. The point cloud is constructed by using the patches of a 512×512 image, which is shown in Figure 7(a). The original image is subsampled and only retains 10% of the pixels at random. The subsampled image is shown in Figure 7(b). One classical problem in image processing is to recover the image from the subsampled image. Here, rather than give an image reconstruction method, we only use this example to demonstrate the performance of the PIM for isotropic elliptic equations.

In this example, the point cloud consists of the patches of the original image. For each pixel x_i in the image f, we extract a patch around it of size 5×5, which is denoted as $p_{x_i}(f)$, where f is the original image. Totally, we get 512^2 patches and each patch is 5×5. The collection of all the patches give a point cloud in $\mathbb{R}^{25}$. Denote this point cloud as $P = \{p_{x_i}(f) : i = 1, \ldots, 512^2\}$. The image is actually corresponding a function u on the point cloud P with $u(p_{x_i}(f)) = f(x_i)$, $f(x_i)$ is the value of image f at pixel x_i. Corresponding to the subsampled image, the value of function u is only known in the patches around the sampled pixels. The collection of all these patches is denoted as S.

Recently, the manifold model has attracted much attention in image processing [30]. In the manifold model, the point cloud P is assumed to be a sample of an underlying manifold, which is called patch manifold. The total variation is used as a regularization to reconstruct the image. The main idea is to minimize the total variation in the patch manifold, i.e.,

$$(4.3) \qquad \min_{u} \|\nabla u\|_{L^1(\mathcal{M})}, \quad \text{subject to:} \quad u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in S.$$

The variation approach tells us that the optimal solution of (4.3) is given by solving following PDE,

$$\text{div} \left(\frac{\nabla u(\mathbf{x})}{|\nabla u(\mathbf{x})|} \right) = 0$$

with the Dirichlet type boundary condition

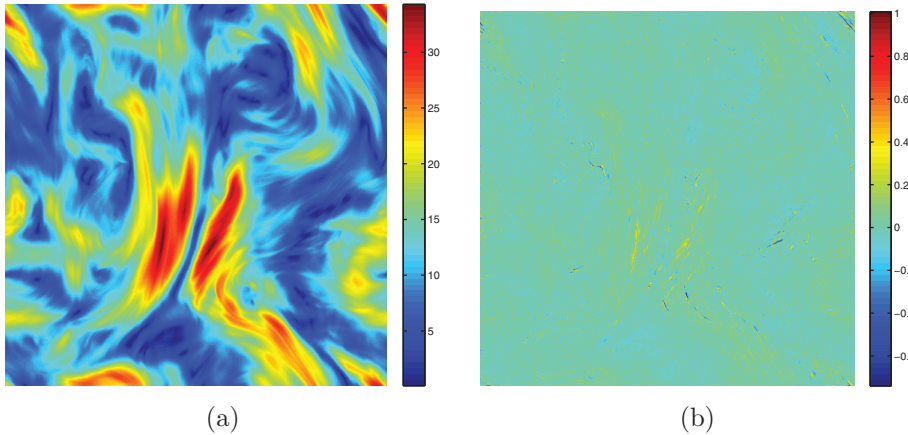$$u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in S.$$

FIG. 8. (a): *reconstructed data;* (b): *residual.*

One natural method to solve the above PDE is an iterative scheme,

$$(4.4) \qquad \mathrm{div}\left(\frac{\nabla u^{n+1}(\mathbf{x})}{|\nabla u^n(\mathbf{x})|}\right) = 0, \qquad u^{n+1}(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in S.$$

In each step, we need to solve an isotropic elliptic equation.

Here, the gradient is computed by using an integral approximation also.

$$\nabla u(\mathbf{x}) = \frac{1}{2t\,\bar{w}_t(\mathbf{x})}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(\mathbf{x}-\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}}$$

$\bar{w}_t(\mathbf{x}) = \int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}$. In the computation, to avoid degeneration of the ellipticity, we regularize the coefficient by adding a small constant in the denominator, i.e., replace $|\nabla u^n(\mathbf{x})|$ by $|\nabla u^n(\mathbf{x})| + \epsilon$ in (4.4) with $\epsilon = 10^{-3}$. The point cloud is assumed to be uniformly distributed, so the volume weight is uniform. The kernel function is a Gaussian function. In this example, we use the integral approximation (2.13) with adaptive $t(\mathbf{x}_i) = \rho(\mathbf{x}_i)^2$, where $\rho(\mathbf{x}_i)$ is the radius of 20 nearest neighbors of $\mathbf{x}_i$.

Figure 8(a) shows the image reconstructed by L_1 extension and Figure 8(b) gives the difference between the original image, Figure 7(a), and the reconstructed image Figure 8(a). As we can see, L_1 extension gives very good reconstruction. This result shows that the PIM solves the isotropic elliptic equation very well on a point cloud.

5. Conclusion. In this paper, we generalize the PIM to solve the isotropic elliptic equation. The PIM is very easy to implement on a point cloud, since it only needs the point cloud without any extra information. Moreover, it also has a very good theoretical property. The coercivity of the original elliptic operator is partially preserved in the PIM. Based on this property, the convergence is proved.

One important implication is the spectral convergence of the point integral method on random samples. Suppose the points are obtained by sampling a manifold according to some probability distribution $p(\mathbf{x})$. In the PIM, the eigenvalue problem

$$(5.1) \qquad \begin{cases} -\frac{1}{p^2(\mathbf{x})}\mathrm{div}(p^2(\mathbf{x})\nabla u(\mathbf{x})) = \lambda u(\mathbf{x}), & \mathbf{x} \in \mathcal{M}, \\ \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = 0, & \mathbf{x} \in \partial\mathcal{M}, \end{cases}$$

is discretized as

$$(5.2) \qquad \frac{1}{t} \sum_{\mathbf{x}_j \in P} R_t(\mathbf{x}_i, \mathbf{x}_j)(u_i - u_j) = \lambda \sum_{\mathbf{x}_j \in P} \bar{R}_t(\mathbf{x}_i, \mathbf{x}_j) u_j.$$

This discretization is closely related with the normalized graph Laplacian. Based on the theoretical results in this paper, it can be proved that the spectra of (5.2) converges to the spectra of (5.1) as the number of sample points goes to infinity.

The other interesting problem is how to generalize the PIM to anisotropic elliptic equation. On this problem, we already have some results. They are going to be reported in a subsequent paper.

REFERENCES

[1] F. ANDREU, J. M. MAZON, J. D. ROSSI, AND J. TOLEDO, *Nonlocal Diffusion Problems*, Math. Surveys Monogr. 165, AMS, Providence, RI, 2010.

[2] R. BARREIRA, C. ELLIOTT, AND A. MADZVAMUSE, *Modelling and simulations of multi-component lipid membranes and open membranes via diffuse interface approaches*, J. Math. Biol., 56 (2008), pp. 347–371.

[3] R. BARREIRA, C. ELLIOTT, AND A. MADZVAMUSE, *The surface finite element method for pattern formation on evolving biological surfaces*, J. Math. Biol., 63 (2011), pp. 1095–1119.

[4] M. BELKIN AND P. NIYOGI, *Laplacian eigenmaps for dimensionality reduction and data representation*, Neural Comput., 15 (2003), pp. 1373–1396.

[5] M. BELKIN AND P. NIYOGI, *Towards a theoretical foundation for Laplacian-based manifold methods*, in Proceedings of COLT, 2005, pp. 486–500.

[6] M. BELKIN AND P. NIYOGI, *Convergence of Laplacian eigenmaps*, Advances in Neural Information Processing Systems, MIT Press, Cambridge, MA, 2008, pp. 1–31

[7] M. BELKIN, J. SUN, AND Y. WANG, *Constructing Laplace operator from point clouds in R^**, in Proceedings of the 19th Annual ACM–SIAM Symposium on Discrete Algorithms (SODA '09), Philadelphia, 2009, SIAM, Philadelphia, pp. 1031–1040.

[8] M. BERTALMIO, L.-T. CHENG, S. OSHER, AND G. SAPIRO, *Variational problems and partial differential equations on implicit surfaces*, J. Comput. Phys., 174 (2001), pp. 759–780.

[9] J. W. CAHN, P. FIFE, AND O. PENROSE, *A phase-field model for diffusion-induced grain-boundary motion*, Ann. Statist., 36 (2008), pp. 555–586.

[10] F. R. K. CHUNG, *Spectral Graph Theory*, AMS, Providence, RI, 1997.

[11] R. R. COIFMAN, S. LAFON, A. B. LEE, M. MAGGIONI, F. WARNER, AND S. ZUCKER, *Geometric diffusions as a tool for harmonic analysis and structure definition of data: Diffusion maps*, in Proc. Natl. Acad. Sci. USA, 2005, pp. 7426–7431.

[12] Q. DU, M. GUNZBURGER, R. B. LEHOUCQ, AND K. ZHOU, *Analysis and approximation of nonlocal diffusion problems with volume constraints*, SIAM Rev., 54 (2012), pp. 667–696.

[13] Q. DU, M. GUNZBURGER, R. B. LEHOUCQ, AND K. ZHOU, *A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws*, Math. Models Methods Appl. Sci., 23 (2013), pp. 493–540.

[14] Q. DU, L. JU, L. TIAN, AND K. ZHOU, *A posteriori error analysis of finite element method for linear nonlocal diffusion and peridynamic models*, Math. Comp., 82 (2013), pp. 1889–1922.

[15] Q. DU, T. LI, AND X. ZHAO, *A convergent adaptive finite element algorithm for nonlocal diffusion and peridynamic models*, SIAM J. Numer. Anal., 51 (2013), pp. 1211–1234.

[16] G. DZIUK AND C. M. ELLIOTT, *Finite element methods for surface PDEs*, Acta Numer., 22 (2013), pp. 289–396.

[17] C. EILKS AND C. M. ELLIOTT, *Numerical simulation of dealloying by surface dissolution via the evolving surface finite element method*, J. Comput. Phys., 227 (2008), pp. 9727–9741.

[18] C. M. ELLIOTT AND B. STINNER, *Modeling and computation of two phase geometric biomembranes using surface finite elements*, J. Comput. Phys., 229 (2010), pp. 6585–6612.

[19] S. GANESAN AND L. TOBISKA, *A coupled arbitrary Lagrangian Eulerian and Lagrangian method for computation of free-surface flows with insoluble surfactants*, J. Comput. Phys., 228 (2009), pp. 2859–2873.

[20] M. HEIN, J.-Y. AUDIBERT, AND U. VON LUXBURG, *From graphs to manifolds—weak and strong pointwise consistency of graph Laplacians*, in Proceedings of the 18th Annual Conference on Learning Theory (COLT '05), 2005. Springer-Verlag, Berlin, pp. 470–485.

[21] A. J. James and J. Lowengrub, *A surfactant-conserving volume-of-fluid method for interfacial flows with insoluble surfactant*, J. Comput. Phys., 201 (2004), pp. 685–722.

[22] S. Lafon, *Diffusion Maps and Geodesic Harmonics*, Ph.D. thesis, Yale University, New Haven, CT, 2004.

[23] R. Lai, J. Liang, and H. Zhao, *A local mesh method for solving PDEs on point clouds*, Inverse Probl. Imaging, 7 (2013), pp. 737–755.

[24] S. Leung, J. Lowengrub, and H. Zhao, *A grid based particle method for solving partial differential equations on evolving surfaces and modeling high order geometrical motion*, J. Comput. Phys., 230 (2011), pp. 2540–2561.

[25] S. Leung and H. Zhao, *A grid based particle method for moving interface problems*, J. Comput. Phys., 228 (2009), pp. 2993–3024.

[26] Z. Li, Z. Shi, and J. Sun, *Point Integral Method for Solving Poisson-Type Equations on Manifolds from Point Clouds with Convergence Guarantees*, preprint, arXiv:1409.2623.

[27] J. Liang and H. Zhao, *Solving partial differential equations on point clouds*, SIAM J. Sci. Comput., 35 (2013), pp. 1461–1486.

[28] C. Macdonald and S. Ruuth, *The implicit closest point method for the numerical solution of partial differential equations on surfaces*, SIAM J. Sci. Comput., 31 (2009), pp. 4330–4350.

[29] M. P. Neilson, J. A. Mackenzie, S. D. Webb, and R. H. Insall, *Modelling cell movement and chemotaxis using pseudopod-based feedback*, SIAM J. Sci. Comput., 33 (2011), pp. 1035–1057.

[30] S. Osher, Z. Shi, and W. Zhu, *Low Dimensional Manifold Model for Image Processing*, Technical report, University of California, Los Angeles, CAM-report 16-04.

[31] G. Peyré, *Manifold models for signals and images*, Comput. Vis. Image Underst., 113 (2009), pp. 248–260.

[32] S. Ruuth and B. Merriman, *A simple embedding method for solving partial differential equations on surfaces*, J. Comput. Phys., 227 (2008), pp. 1943–1961.

[33] Z. Shi and J. Sun, *Convergence of the Point Integral Method for the Poisson Equation on Manifolds* I: *The Neumann Boundary*, preprint, arXiv:1403.2141.

[34] Z. Shi and J. Sun, *Convergence of the Point Integral Method for the Poisson Equation on Manifolds* II: *The Dirichlet Boundary*, preprint, arXiv:1312.4424.

[35] A. Singer, *From graph to manifold Laplacian: The convergence rate*, Appl. Comput. Harmon. Anal., 21 (2006), pp. 128–134.

[36] A. Singer and H. tieng Wu, *Spectral Convergence of the Connection Laplacian from Random Samples*, preprint, arXiv:1306.1587.

[37] J. Xu and H. Zhao, *An Eulerian formulation for solving partial differential equations along a moving interface*, J. Sci. Comput., 19 (2003), pp. 573–594.

[38] K. Zhou and Q. Du, *Mathematical and numerical analysis of linear peridynamic models with nonlocal boundary conditions*, SIAM J. Numer. Anal., 48 (2010), pp. 1759–1780.