Convergence of the Point Integral method for Poisson equation on point cloud

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Abstract

The Laplace-Beltrami operator (LBO) is a fundamental object associated to Riemannian manifolds, which encodes all intrinsic geometry of the manifolds and has many desirable properties. Recently, we proposed a novel numerical method, Point Integral method (PIM), to discretize the Laplace-Beltrami operator on point clouds [28]. In this paper, we analyze the convergence of Point Integral method (PIM) for Poisson equation with Neumann boundary condition on submanifolds isometrically embedded in Euclidean spaces.

1 Introduction

The partial differential equations on manifolds arise in a wide variety of applications. In many problems, including material science [10, 19], fluid flow [21, 23], biology and biophysics [3, 20, 32, 2], people need to study the physical process, for instance diffusion and convection, in curved surfaces which introduce different kinds of PDEs in surfaces. It has been several decades to develop numerical methods for solving PDEs in surfaces. Many methods have been developed, such as surface finite element method [18], level set method [9, 40], grid based particle method [27, 26] and closest point method [36, 31].

Recently, manifold model attracts more and more attentions in data analysis and image processing [35, 34, 4, 12]. It is well known that PDEs on the manifold, especially the Laplace equation, encode lots of intrinsic information of the manifold which is very helpful to reveal the underlying structure hidden in the data. In the data analysis problems, data is usually represented as a collection of points embedding in a high dimensional Euclidean space, which is refereed as point cloud. The point cloud gives a sample of the manifold and we need to solve PDEs on the unstructured point cloud. Usually, the point cloud is embedded in a high dimensional space, the traditional methods for PDEs on 2D surfaces do not work in this case.

In past few years, many efforts were devoted to develop alternative numerical methods to discretize the differential operators on point cloud. Liang et al. proposed to discretize the differential operators on point cloud by local least square approximations of the manifold [29]. Their method can achieve high order accuracy and enjoy more flexibility since no mesh is needed. In principle, it can be applied to manifolds with arbitrary dimensions and co-dimensions with or without boundary. However, if the dimension of the manifold is high, this method may not be stable since high order polynomial is used to fit the data. Later, Lai et al. proposed local mesh method to approximate the differential operators on point cloud [25]. The main idea is to construct mesh locally around each point by using K nearest neighbors. The local mesh is easier to construct than global mesh. Based on the local mesh, it is easy to discretize differential operators and compute integrals. However, when the dimension of the manifold is high, even local mesh is not easy to construct.

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In [28], we have proposed a novel numerical method, point integral method (PIM), to solve the Poisson equation on point cloud. The main idea of the point integral method is to approximate the Poisson equation by the following integral equation:

$$-\int_{\mathcal{M}} \Delta_{\mathcal{M}} u(\mathbf{y}) \bar{R}_t(\mathbf{x}, \mathbf{y}) d\mu_{\mathbf{y}} \approx \frac{1}{t} \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y})(u(\mathbf{x}) - u(\mathbf{y})) d\mu_{\mathbf{y}} - 2 \int_{\partial \mathcal{M}} \bar{R}_t(\mathbf{x}, \mathbf{y}) \frac{\partial u}{\partial \mathbf{n}}(\mathbf{y}) d\tau_{\mathbf{y}}, \quad (1.1)$$

where $\mathbf{n}$ is the out normal of $\mathcal{M}$, $\mathcal{M}$ is a smooth k-dimensional manifold embedded in $\mathbb{R}^d$ and $\partial \mathcal{M}$ is the boundary of $\mathcal{M}$. $R_t(\mathbf{x}, \mathbf{y})$ and $\bar{R}_t(\mathbf{x}, \mathbf{y})$ are kernel functions given as follows

$$R_t(\mathbf{x}, \mathbf{y}) = C_t R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t}\right), \quad \bar{R}_t(\mathbf{x}, \mathbf{y}) = C_t \bar{R}\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t}\right) \quad (1.2)$$

where $C_t = \frac{1}{(4\pi t)^{k/2}}$ is the normalizing factor. $R \in C^2(\mathbb{R}^+)$ be a positive function which is integrable over $[0, +\infty)$,

$$\bar{R}(r) = \int_r^{+\infty} R(s) ds.$$

$\Delta_{\mathcal{M}} = \mathrm{div}(\nabla)$ is the Laplace-Beltrami operator on $\mathcal{M}$. Let $\Phi : \Omega \subset \mathbb{R}^k \to \mathcal{M} \subset \mathbb{R}^d$ be a local parametrization of $\mathcal{M}$ and $\theta \in \Omega$. For any differentiable function $f : \mathcal{M} \to \mathbb{R}$, define the gradient on the manifold

$$\nabla f(\Phi(\theta)) = \sum_{i,j=1}^m g^{ij}(\theta) \frac{\partial \Phi}{\partial \theta_i}(\theta) \frac{\partial f(\Phi(\theta))}{\partial \theta_j}(\theta), \quad (1.3)$$

and for vector field $F : \mathcal{M} \to T_{\mathbf{x}}\mathcal{M}$ on $\mathcal{M}$, where $T_{\mathbf{x}}\mathcal{M}$ is the tangent space of $\mathcal{M}$ at $\mathbf{x} \in \mathcal{M}$, the divergence is defined as

$$\mathrm{div}(F) = \frac{1}{\sqrt{\det G}} \sum_{k=1}^d \sum_{i,j=1}^m \frac{\partial}{\partial \theta_i} \left(\sqrt{\det G} g^{ij} F^k(\Phi(\theta)) \frac{\partial \Phi^k}{\partial \theta_j}\right) \quad (1.4)$$

where $(g^{ij})_{i,j=1,\cdots,k} = G^{-1}$, $\det G$ is the determinant of matrix G and $G(\theta) = (g_{ij})_{i,j=1,\cdots,k}$ is the first fundamental form which is defined by

$$g_{ij}(\theta) = \sum_{k=1}^d \frac{\partial \Phi_k}{\partial \theta_i}(\theta) \frac{\partial \Phi_k}{\partial \theta_j}(\theta), \quad i,j = 1, \cdots, m. \quad (1.5)$$

and $(F^1(\mathbf{x}), \cdots, F^d(\mathbf{x}))^t$ is the representation of F in the embedding coordinates.

Using the integral approximation, we transfer the Laplace-Beltrami operator to an integral operator. The integral operator is easy to be discretized on point clouds using some quadrature rule, since there is not any differential operators inside. This is the essential ingredient in the point integral method. Similar integral approximation is also used in nonlocal diffusion and peridynamic model [14, 1, 15, 16, 41].

The point integral method is also related with the graph Laplacian. Graph Laplacian is a discrete object associated to a graph, which reveals many properties of graphs [11]. It is observed in [5, 24, 22, 37] that the graph Laplacian with the Gaussian weights well approximates the LBO when the vertices of the graph are assumed to sample the underlying manifold. When there is no boundary, Belkin and Niyogi [6] showed the spectra of the graph Laplacian with Gaussian weights converges to that of $\Delta_{\mathcal{M}}$. The main issue that remains is how to deal with the boundary. In fact, near the boundary, it was observed [24, 7] that the graph Laplacian is dominated by the first order derivative and thus fails to be true Laplacian. Recently, Singer and Wu [38] showed the spectral convergence of the graph Laplacian in the presence of the Neumann boundary. In both [6] and [38],

the convergence analysis is based on the connection between the graph Laplacian and the heat operator, and thus the Gaussian weights are essential.

The main contribution of this paper is that, for Poisson equation with Neumann boundary condition, we prove that the numerical solution computed by the PIM converges to the exact solution in H^1 norm as the density of the sample points tends to infinity. Unlike the methods used in graph Laplacian, we do not relate the integral operator to heat kernel. Instead, we use the strategy which is standard in numerical analysis to prove the convergence.

It is well known in the numerical analysis that the convergence is the summation of consistency and stability. We prove that the coercivity of the original Laplace-Beltrami operator is preserved in the point integral method. This implies the stability of the point integral method. Together with the estimate of the truncation error, we get the convergence of the point integral method.

The remaining of this paper is organized as following. In Section 2, we describe the point integral method for Poisson equation with Neumann boundary condition. The convergence result is stated in Section 3. The structure of the proof is shown in Section 4. The main body of the proof is in Section 5, Section 6 and Section 7. Finally, conclusions and discussion on the future work are given in Section 8.

2   Point Integral Method

In this paper, we consider Poisson equation on a smooth, compact k-dimensional submanifold $\mathcal{M}$ in $\mathbb{R}^d$, $d \geq k$ with the Neumann boundary

$$\begin{cases} \Delta_{\mathcal{M}} u(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x} \in \mathcal{M} \\ \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = b(\mathbf{x}), & \mathbf{x} \in \partial\mathcal{M} \end{cases} \tag{2.1}$$

The manifold $\mathcal{M}$ is sampled with a set of sample points P and a subset $S \subset P$ sampling the boundary of $\mathcal{M}$. List the points in P respectively S in a fixed order $P = (\mathbf{p}_1, \cdots, \mathbf{p}_n)$ where $\mathbf{p}_i \in \mathbb{R}^d, 1 \leq i \leq n$, respectively $S = (\mathbf{s}_1, \cdots, \mathbf{s}_m) \subset P$.

In addition, assume we are given two vectors $\mathbf{V} = (V_1, \cdots, V_n)^t$ where V_i is an volume weight of $\mathbf{p}_i$ in $\mathcal{M}$, and $\mathbf{A} = (A_1, \cdots, A_m)^t$ where A_i is an area weight of $\mathbf{s}_i$ in $\partial\mathcal{M}$, so that for any $f \in C^1(\mathcal{M})$ and $g \in C^1(\mathcal{M})$,

$$\sum_{i=1}^n f(\mathbf{p}_i)V_i \approx \int_{\mathcal{M}} f(\mathbf{x})d\mu_{\mathbf{x}}, \quad \sum_{i=1}^m g(\mathbf{s}_i)A_i \approx \int_{\partial\mathcal{M}} g(\mathbf{x})d\tau_{\mathbf{x}}.$$

Here $d\mu_{\mathbf{x}}$ and $d\tau_{\mathbf{x}}$ are the volume form of $\mathcal{M}$ and $\partial\mathcal{M}$, respectively.

Using the integral approximation (1.1), the Poisson equation is approximated by an integral equation,

$$\frac{1}{t} \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y})(u(\mathbf{x}) - u(\mathbf{y}))d\mu_{\mathbf{y}} - 2\int_{\partial\mathcal{M}} \bar{R}_t(\mathbf{x}, \mathbf{y})b(\mathbf{y})d\tau_{\mathbf{y}} = \int_{\mathcal{M}} f(\mathbf{y})\bar{R}_t(\mathbf{x}, \mathbf{y})d\mu_{\mathbf{y}} \tag{2.2}$$

In the integral equation, there is not any differential operators. It is easy to discretize on the point cloud with the weight vectors, $\mathbf{V}$ and $\mathbf{A}$,

$$\sum_{\mathbf{p}_j \in P} R_t(\mathbf{p}_i, \mathbf{p}_j)(u_i - u_j)V_j - 2\sum_{\mathbf{s}_j \in S} \bar{R}_t(\mathbf{p}_i, \mathbf{s}_j)b(\mathbf{s}_j)A_j = \sum_{\mathbf{p}_j \in P} \bar{R}_t(\mathbf{p}_i, \mathbf{p}_j)f(\mathbf{p}_j)V_j \tag{2.3}$$

The solution $\mathbf{u} = (u_1, \cdots, u_n)^t$ to above linear system gives an approximation of the solution to the problem (2.1).

Remark 2.1. *In the point integral method, we need the volume weight $\mathbf{V}$ and area weight $\mathbf{A}$. We remark that it is much easier to obtain the volume weight $\mathbf{V}$ than to generate a consistent mesh for $\mathcal{M}$ with good shaped elements. If $\mathbf{V}$ and $\mathbf{A}$ are not given, they can be estimated as follows.*

- *If a mesh with the vertices P approximating $\mathcal{M}$ is given, both weight vectors $\mathbf{V}$ and $\mathbf{A}$ can be easily estimated from the given mesh by summing up the volume of the simplices incident to the vertices. Note that there is no requirement on the shape of the elements in the mesh.*

- *If the points in P and S are independent samples from some distribution on $\mathcal{M}$ and $\partial\mathcal{M}$ respectively, then $\mathbf{V}$ and $\mathbf{A}$ can be obtained from the distribution.*

- *Finally, following [30], one can estimate the vectors $\mathbf{V}$ and $\mathbf{A}$ by locally approximating tangent spaces of $\mathcal{M}$ and $\partial\mathcal{M}$, respectively. Specifically, for a point $\mathbf{p} \in P$, project the samples near to $\mathbf{p}$ in P onto the approximated tangent space at $\mathbf{p}$ and take the volume of the Voronoi cell of $\mathbf{p}$ as its weight. In this way, one avoids constructing globally consistent meshes for $\mathcal{M}$ and $\partial\mathcal{M}$.*

3   Main Results

The main contribution in this paper is to establish the convergence results for the point integral method for solving the problem (2.1). To simplify the notation and make the proof concise, in the analysis, we consider the homogeneous Neumann boundary conditions, i.e.

$$\begin{cases} -\Delta_{\mathcal{M}} u(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x} \in \mathcal{M} \\ \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = 0, & \mathbf{x} \in \partial\mathcal{M} \end{cases} \tag{3.1}$$

The analysis can be easily generalized to the non-homogeneous boundary conditions.

The corresponding numerical scheme is

$$\frac{1}{t} \sum_{\mathbf{p}_j \in P} R_t(\mathbf{p}_i, \mathbf{p}_j)(u_i - u_j)V_j = \sum_{\mathbf{p}_j \in P} \bar{R}_t(\mathbf{p}_i, \mathbf{p}_j)f_j V_j. \tag{3.2}$$

where $f_j = f(\mathbf{p}_j)$.

Before proving the convergence of the point integral method, we need to clarify the meaning of the convergence between the point cloud $(P, \mathbf{V})$ and the manifold $\mathcal{M}$. In this paper, we consider the convergence in the sense that $h(P, \mathbf{V}, \mathcal{M}) \to 0$ where $h(P, \mathbf{V}, \mathcal{M})$ is the *integral accuracy index* defined as following,

Definition 3.1 (Integral Accuracy Index). *For the point cloud $(P, \mathbf{V})$ which samples the manifold $\mathcal{M}$, the integral accuracy index $h(P, \mathbf{V}, \mathcal{M})$ is defined as*

$$h(P, \mathbf{V}, \mathcal{M}) = \sup_{f \in C^1(\mathcal{M})} \frac{\left| \int_{\mathcal{M}} f(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} - \sum_{\mathbf{p}_i \in P} f(\mathbf{p}_i)V_i \right|}{|\mathrm{supp}(f)| \|f\|_{C^1(\mathcal{M})}}.$$

where $\|f\|_{C^1(\mathcal{M})} = \|f\|_\infty + \|\nabla f\|_\infty$ and $|\mathrm{supp}(f)|$ is the volume of the support of f.

Using the definition of integrable index, we say that the point cloud $(P, \mathbf{V})$ converges to the manifold $\mathcal{M}$ if $h(P, \mathbf{V}, \mathcal{M}) \to 0$. In the convergence analysis, we assume that $h(P, \mathbf{V}, \mathcal{M})$ is small enough.

Remark 3.1. *In some sense, $h(P, \mathbf{V}, \mathcal{M})$ is a measure of the density of the point cloud.*

- *If the volume weight $\mathbf{V}$ comes from a mesh, one can obtain the integral accuracy index $h(P, \mathbf{V}, \mathcal{M}) = O(\rho)$ where ρ is the size of the elements in the mesh and the angle between the normal space of an element and the normal space of $\mathcal{M}$ at the vertices of the element is of order $\rho^{1/2}$ [39].*

- *If the point cloud is sampled from some distribution, from central limit theorem, $h(P, \mathbf{V}, \mathcal{M}) \sim O(1/\sqrt{n})$ where n is the number of point in P.*

Remark 3.2. *To consider the non-homogeneous Neumann boundary condition or Dirichlet boundary condition, we have to also assume that $h(S, \mathbf{A}, \partial\mathcal{M}) \to 0$, where S is the point set sample the boundary $\partial\mathcal{M}$ and $\mathbf{A}$ is the corresponding volume weight on the boundary $\partial\mathcal{M}$.*

To get the convergence, we also need some assumptions on the regularity of the submanifold $\mathcal{M}$ and the integral kernel function R.

Assumption 3.1.

- Smoothness of the manifold: $\mathcal{M}, \partial\mathcal{M}$ are both compact and C^∞ smooth k-dimensional submanifolds isometrically embedded in a Euclidean space $\mathbb{R}^d$.

- Assumptions on the kernel function $R(r)$:

  (a) Smoothness: $R \in C^2(\mathbb{R}^+)$;

  (b) Nonnegativity: $R(r) \geq 0$ for any $r \geq 0$.

  (c) Compact support: $R(r) = 0$ for $\forall r > 1$;

  (d) Nondegeneracy: $\exists \delta_0 > 0$ so that $R(r) \geq \delta_0$ for $0 \leq r \leq \frac{1}{2}$.

Remark 3.3. *The assumption on the kernel function is very mild. The compact support assumption can be relaxed to exponentially decay, like Gaussian kernel. In the nondegeneracy assumption, $1/2$ may be replaced by a positive number θ_0 with $0 < \theta_0 < 1$. Similar assumptions on the kernel function is also used in analysis the nonlocal diffusion problem [17].*

Remark 3.4. *It is for the sake of simplicity that R is assumed to be compactly supported. After some mild modifications of the proof, the same convergence results also hold for any kernel function that decays exponentially, like the Gaussian kernel $G_t(\mathbf{x}, \mathbf{y}) = C_t \exp\left(-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right)$. In fact, for any $s \geq 1$ and any $\epsilon > 0$, the H^s mass of the Gaussian kernel over the domain $\Omega = \{\mathbf{y} \in \mathcal{M} | |\mathbf{x} - \mathbf{y}|^2 \geq t^{1+\epsilon}\}$ decays faster than any polynomial in t as t goes to 0, i.e., $\lim_{t \to 0} \frac{\|G_t(\mathbf{x}, \mathbf{y})\|_{H^s(\Omega)}}{t^\alpha} = 0$ for any α. In this way, we can bound any influence of the integral outside a compact support.*

All the analysis in this paper is under the assumptions in Assumption 3.1 and $h(P, \mathbf{V}, \mathcal{M})$, t are small enough. In the theorems and the proof, without introducing any confusions, we omit the statement of the assumptions.

The solution of the point integral method is a vector $\mathbf{u}$ while the solution of the problem (3.1) is a function defined on $\mathcal{M}$. To make them comparable, for any solution $\mathbf{u} = (u_1, \cdots, u_n)^t$ to the problem (3.2), we construct a function on $\mathcal{M}$

$$I_{\mathbf{f}}(\mathbf{u})(\mathbf{x}) = \frac{\sum_{\mathbf{p}_j \in P} R_t(\mathbf{x}, \mathbf{p}_j) u_j V_j - t \sum_{\mathbf{p}_j \in P} \bar{R}_t(\mathbf{x}, \mathbf{p}_j) f_j V_j}{\sum_{\mathbf{p}_j \in P} R_t(\mathbf{x}, \mathbf{p}_j) V_j}. \tag{3.3}$$

It is easy to verify that $I_{\mathbf{f}}(\mathbf{u})$ interpolates $\mathbf{u}$ at the sample points P, i.e., $I_{\mathbf{f}}(\mathbf{u})(\mathbf{p}_j) = u_j$ for any $\mathbf{p}_j \in P$. The following theorem guarantees the convergence of the point integral method.

Theorem 3.1. *Let u be the solution to Problem (3.1) with $f \in C^1(\mathcal{M})$ and the vector $\mathbf{u}$ be the solution to the problem (3.2). Then there exists constants C and T_0 only depend on $\mathcal{M}$, such that for any $t \leq T_0$*

$$\|u - I_{\mathbf{f}}(\mathbf{u})\|_{H^1(\mathcal{M})} \leq C \left(t^{1/2} + \frac{h(P, \mathbf{V}, \mathcal{M})}{t^{3/2}}\right) \|f\|_{C^1(\mathcal{M})}. \tag{3.4}$$

where $h(P, \mathbf{V}, \mathcal{M})$ is the integral accuracy index.

4   Structure of the Proof

To simplify the notation, we introduce an integral operator,

$$L_t u = \frac{1}{t} \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y})(u(\mathbf{x}) - u(\mathbf{y})) \mathrm{d}\mu_{\mathbf{y}} \tag{4.1}$$

Roughly speaking, the proof the convergence includes estimate of the truncation error $L_t(u - I_{\mathbf{f}}(\mathbf{u}))$ and the stability of the integral operator L_t. Here $u(\mathbf{x})$ is the solution of the problem (3.1) and $\mathbf{u}$ is the solution of the problem (3.2).

First, we have following theorem regarding the stability of the operator L_t.

Theorem 4.1. *Let $u(\mathbf{x})$ solves the integral equation*

$$L_t u = r(\mathbf{x})$$

where $r \in H^1(\mathcal{M})$ with $\int_{\mathcal{M}} r(\mathbf{x}) \mathrm{d}\mu_{\mathbf{x}} = 0$. There exist constants $C > 0, T_0 > 0$ independent on t, such that

$$\|u\|_{H^1(\mathcal{M})} \le C \left(\|r\|_{L^2(\mathcal{M})} + t \|\nabla r\|_{L^2(\mathcal{M})} \right)$$

as long as $t \le T_0$.

To apply the stability result, we need L_2 estimate of $L_t(u - I_{\mathbf{f}}(\mathbf{u}))$ and $\nabla L_t(u - I_{\mathbf{f}}(\mathbf{u}))$. These truncation errors are analyzed in following two theorems by spliting the truncation error $L_t(u - I_{\mathbf{f}}(\mathbf{u}))$

$$L_t(u - I_{\mathbf{f}}(\mathbf{u})) = L_t(u - u_t)) + L_t(u_t - I_{\mathbf{f}}(\mathbf{u}))$$

where u_t is the solution of the integral equation

$$\frac{1}{t} \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y})(u(\mathbf{x}) - u(\mathbf{y})) \mathrm{d}\mu_{\mathbf{y}} = \int_{\mathcal{M}} f(\mathbf{y}) \bar{R}_t(\mathbf{x}, \mathbf{y}) \mathrm{d}\mu_{\mathbf{y}}. \tag{4.2}$$

For the second term, we have

Theorem 4.2. *Let $u_t(\mathbf{x})$ be the solution of the problem (4.2) and $\mathbf{u}$ be the solution of the problem (3.2). If $f \in C^1(\mathcal{M})$, then there exists constants C, T_0 depending only on $\mathcal{M}$, so that*

$$\|L_t (I_{\mathbf{f}} \mathbf{u} - u_t)\|_{L^2(\mathcal{M})} \le \frac{Ch(P, \mathbf{V}, \mathcal{M})}{t^{3/2}} \|f\|_{C^1(\mathcal{M})}, \tag{4.3}$$

$$\|\nabla L_t (I_{\mathbf{f}} \mathbf{u} - u_t)\|_{L^2(\mathcal{M})} \le \frac{Ch(P, \mathbf{V}, \mathcal{M})}{t^2} \|f\|_{C^1(\mathcal{M})}. \tag{4.4}$$

as long as $t \le T_0$ and $\frac{h(P, \mathbf{V}, \mathcal{M})}{\sqrt{t}} \le T_0$, $h(P, \mathbf{V}, \mathcal{M})$ is the integral difference index in Definition 3.1.

In the analysis, we found that the error term $L_t(u - u_t)$ has boundary layer structure. In the interior region, it is $O(\sqrt{t})$ and in a layer adjacent to the boundary with width $O(\sqrt{t})$, the error is $O(1)$.

Theorem 4.3. *Let $u(\mathbf{x})$ be the solution of the problem (3.1) and $u_t(\mathbf{x})$ be the solution of the corresponding integral equation (4.2). Let*

$$I_{bd} = \sum_{j=1}^{d} \int_{\partial\mathcal{M}} n^j(\mathbf{y})(\mathbf{x} - \mathbf{y}) \cdot \nabla(\nabla^j u(\mathbf{y})) \bar{R}_t(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}}, \tag{4.5}$$

and

$$L_t(u - u_t) = I_{in} + I_{bd}.$$

where $\mathbf{n}(\mathbf{y}) = (n^1(\mathbf{y}), \cdots, n^d(\mathbf{y}))$ *is the out normal vector of* $\partial\mathcal{M}$ *at* $\mathbf{y}$, ∇^j *is the jth component of gradient* ∇.

If $u \in H^3(\mathcal{M})$, *then there exists constants* C, T_0 *depending only on* $\mathcal{M}$ *and* $p(\mathbf{x})$, *so that,*

$$\|I_{in}\|_{L^2(\mathcal{M})} \leq Ct^{1/2}\|u\|_{H^3(\mathcal{M})}, \quad \|\nabla I_{in}\|_{L^2(\mathcal{M})} \leq C\|u\|_{H^3(\mathcal{M})}, \tag{4.6}$$

as long as $t \leq T_0$.

To utilizing the boundary layer structure, we need a stability result specifically for the boundary term.

Theorem 4.4. *Let* $u(\mathbf{x})$ *solves the integral equation*

$$L_t u = \int_{\partial\mathcal{M}} \mathbf{b}(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})\bar{R}_t(\mathbf{x}, \mathbf{y})\mathrm{d}\tau_{\mathbf{y}} - \bar{b}$$

where $|\mathcal{M}| = \int_{\mathcal{M}} \mathrm{d}\mu_{\mathbf{x}}$ *and*

$$\bar{b} = \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} \left(\int_{\partial\mathcal{M}} \mathbf{b}(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})\bar{R}_t(\mathbf{x}, \mathbf{y})\mathrm{d}\tau_{\mathbf{y}} \right) \mathrm{d}\mathbf{x}.$$

Then, there exist constant $C > 0, T_0 > 0$ *independent on* t, *such that*

$$\|u\|_{H^1(\mathcal{M})} \leq C\sqrt{t} \, \|\mathbf{b}\|_{H^1(\mathcal{M})}.$$

as long as $t \leq T_0$.

Theorem 3.1 is an easy corollary from Theorems 4.1, 4.2, 4.3 and 4.4. Theorem 4.2 and Theorem 4.1 imply that

$$\|u_t - I_{\mathbf{f}}(\mathbf{u})\|_{H^1(\mathcal{M})} = O\left(\frac{h(P, \mathbf{V}, \mathcal{M})}{t^{3/2}} \right).$$

and Theorem 4.1, 4.3 and 4.4 imply

$$\|u - u_t\|_{H^1(\mathcal{M})} = O\left(t^{1/2} \right),$$

which prove Theorem 3.1.

In the rest of the paper, we prove Theorem 4.1, 4.2, 4.3 and 4.4 respectively.

5 Error analysis of the integral approximation (Theorem 4.3)

In this section, we need to introduce a special parametrization of the manifold $\mathcal{M}$. This parametrization is based on following proposition.

Proposition 5.1. *Assume both* $\mathcal{M}$ *and* $\partial\mathcal{M}$ *are* C^2 *smooth and* σ *is the minimum of the reaches of* $\mathcal{M}$ *and* $\partial\mathcal{M}$. *For any point* $\mathbf{x} \in \mathcal{M}$, *there is a neighborhood* $U \subset \mathcal{M}$ *of* $\mathbf{x}$, *so that there is a parametrization* $\Phi : \Omega \subset \mathbb{R}^k \to U$ *satisfying the following conditions. For any* $\rho \leq 0.1$,

    (i) Ω *is convex and contains at least half of the ball* $B_{\Phi^{-1}(\mathbf{x})}(\frac{\rho}{5}\sigma)$, *i.e.,* $vol(\Omega \cap B_{\Phi^{-1}(\mathbf{x})}(\frac{\rho}{5}\sigma)) > \frac{1}{2}(\frac{\rho}{5}\sigma)^k w_k$ *where* w_k *is the volume of unit ball in* $\mathbb{R}^k$;

    (ii) $B_{\mathbf{x}}(\frac{\rho}{10}\sigma) \cap \mathcal{M} \subset U$.

(iii) The determinant the Jacobian of Φ is bounded: $(1 - 2\rho)^k \leq |D\Phi| \leq (1 + 2\rho)^k$ over Ω.

(iv) For any points $\mathbf{y}, \mathbf{z} \in U$, $1 - 2\rho \leq \frac{|\mathbf{y} - \mathbf{z}|}{|\Phi^{-1}(\mathbf{y}) - \Phi^{-1}(\mathbf{z})|} \leq 1 + 3\rho$.

This proposition basically says there exists a local parametrization of small distortion if $(\mathcal{M}, \partial\mathcal{M})$ satisfies certain smoothness, and moreover, the parameter domain is convex and big enough. The proof of this proposition can be found in Appendix A. Next, we introduce a special parametrization of the manifold $\mathcal{M}$.

Let $\rho = 0.1$, σ be the minimum of the reaches of $\mathcal{M}$ and $\partial\mathcal{M}$ and $\delta = \rho\sigma/20$. For any $\mathbf{x} \in \mathcal{M}$, denote

$$B_{\mathbf{x}}^{\delta} = \{\mathbf{y} \in \mathcal{M} : |\mathbf{x} - \mathbf{y}| \leq \delta\}, \quad \mathcal{M}_{\mathbf{x}}^t = \{\mathbf{y} \in \mathcal{M} : |\mathbf{x} - \mathbf{y}|^2 \leq 4t\} \tag{5.1}$$

and we assume t is small enough such that $2\sqrt{t} \leq \delta$.

Since the manifold $\mathcal{M}$ is compact, there exists a δ-net, $\mathcal{N}_\delta = \{\mathbf{q}_i \in \mathcal{M}, \ i = 1, \cdots, N\}$, such that

$$\mathcal{M} \subset \bigcup_{i=1}^{N} B_{\mathbf{q}_i}^{\delta}.$$

and there exists a partition of $\mathcal{M}$, $\{\mathcal{O}_i, \ i = 1, \cdots, N\}$, such that $\mathcal{O}_i \cap \mathcal{O}_j = \emptyset$, $i \neq j$ and

$$\mathcal{M} = \bigcup_{i=1}^{N} \mathcal{O}_i, \quad \mathcal{O}_i \subset B_{\mathbf{q}_i}^{\delta}, \quad i = 1, \cdots, N.$$

Using Proposition 5.1, there exist a parametrization $\Phi_i : \Omega_i \subset \mathbb{R}^k \to U_i \subset \mathcal{M}$, $i = 1, \cdots, N$, such that

1. (Convexity) $B_{\mathbf{q}_i}^{2\delta} \subset U_i$ and Ω_i is convex.

2. (Smoothness) $\Phi_i \in C^3(\Omega_i)$;

3. (Locally small deformation) For any points $\theta_1, \theta_2 \in \Omega_i$,

$$\frac{1}{2}|\theta_1 - \theta_2| \leq \|\Phi_i(\theta_1) - \Phi_i(\theta_2)\| \leq 2|\theta_1 - \theta_2|.$$

Using the partition, $\{\mathcal{O}_i, \ i = 1, \cdots, N\}$, for any $\mathbf{y} \in \mathcal{M}$, there exists unique $J(\mathbf{y}) \in \{1, \cdots, N\}$, such that

$$\mathbf{y} \in \mathcal{O}_{J(\mathbf{y})} \subset B_{\mathbf{q}_{J(\mathbf{y})}}^{\delta}. \tag{5.2}$$

Moreover, using the condition, $2\sqrt{t} \leq \delta$, we have $\mathcal{M}_{\mathbf{y}}^t \subset B_{\mathbf{q}_{J(\mathbf{y})}}^{2\delta} \subset U_{J(\mathbf{y})}$. Then $\Phi_{J(\mathbf{y})}^{-1}(\mathbf{x})$ and $\Phi_{J(\mathbf{y})}^{-1}(\mathbf{y})$ are both well defined for any $\mathbf{x} \in \mathcal{M}_{\mathbf{y}}^t$.

Now, we define an auxiliary function, $\eta(\mathbf{x}, \mathbf{y})$ for any $\mathbf{y} \in \mathcal{M}$, $\mathbf{x} \in \mathcal{M}_{\mathbf{y}}^t$. Let

$$\xi(\mathbf{x}, \mathbf{y}) = \Phi_{J(\mathbf{y})}^{-1}(\mathbf{x}) - \Phi_{J(\mathbf{y})}^{-1}(\mathbf{y}) \in \mathbb{R}^k, \quad \eta(\mathbf{x}, \mathbf{y}) = \xi(\mathbf{x}, \mathbf{y}) \cdot \partial\Phi_{J(\mathbf{y})}(\alpha(\mathbf{x}, \mathbf{y})) \in \mathbb{R}^d, \tag{5.3}$$

where $\alpha(\mathbf{x}, \mathbf{y}) = \Phi_{J(\mathbf{y})}^{-1}(\mathbf{y})$ and ∂ is the gradient operator in the parameter space, i.e.

$$\partial\Phi_j(\theta) = \left(\frac{\partial\Phi_j}{\partial\theta_1}(\theta), \frac{\partial\Phi_j}{\partial\theta_2}(\theta), \cdots, \frac{\partial\Phi_j}{\partial\theta_k}(\theta)\right), \quad \theta \in \Omega_j \subset \mathbb{R}^k.$$

Now we state the proof of Theorem 4.3.

Proof. Let $r(\mathbf{x}) = -(L_t u - L_t u_t)$ be the residual, then we have

$$
\begin{aligned}
r(\mathbf{x}) &= -\frac{1}{t}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}} + 2\int_{\partial\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})g(\mathbf{y})\mathrm{d}\tau_{\mathbf{y}} - \int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})f(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\
&= -\frac{1}{t}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}} + \int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})\Delta_{\mathcal{M}}u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\
&\quad + \frac{1}{t}\int_{\mathcal{M}} (\mathbf{x}-\mathbf{y})\cdot\nabla u(\mathbf{y})R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathrm{d}\mu_{\mathbf{y}} \\
&= -\frac{1}{t}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y})-(\mathbf{x}-\mathbf{y})\cdot\nabla u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}} + \int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})\Delta_{\mathcal{M}}u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}.
\end{aligned}
$$

Here we use that fact that

$$
\int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})f(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} = \int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})\Delta_{\mathcal{M}}u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}},
$$

and

$$
\begin{aligned}
\int_{\partial\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})g(\mathbf{y})\mathrm{d}\tau_{\mathbf{y}} &= \int_{\partial\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})\frac{\partial u}{\partial\mathbf{n}}(\mathbf{y})\mathrm{d}\tau_{\mathbf{y}} \\
&= \int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})\Delta_{\mathcal{M}}u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} + \int_{\mathcal{M}} \nabla_{\mathbf{y}}\bar{R}_t(\mathbf{x},\mathbf{y})\cdot\nabla u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\
&= \int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})\Delta_{\mathcal{M}}u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} + \frac{1}{2t}\int_{\mathcal{M}} (\mathbf{x}-\mathbf{y})\cdot\nabla u(\mathbf{y})R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}},
\end{aligned}
$$

where the last equality comes from:

$$
\begin{aligned}
&\int_{\mathcal{M}} \nabla u(\mathbf{y})\cdot\nabla_{\mathbf{y}}\bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} && (5.4) \\
&= \frac{1}{2t}\int_{\mathcal{M}} \left(\partial_{i'}\Phi^l g^{i'j'}\partial_{j'}u(\mathbf{y})\right)\left(\partial_{m'}\Phi^l g^{m'n'}\partial_{n'}\Phi^j(x^j-y^j)R_t(\mathbf{x},\mathbf{y})\right)\mathrm{d}\mu_{\mathbf{y}} \\
&= \frac{1}{2t}\int_{\mathcal{M}} \left(\partial_{n'}\Phi^j g^{j'n'}\partial_{j'}u(\mathbf{y})\right)\left((x^j-y^j)R_t(\mathbf{x},\mathbf{y})\right)\mathrm{d}\mu_{\mathbf{y}} \\
&= \frac{1}{2t}\int_{\mathcal{M}} (x^j-y^j)\nabla^j u(\mathbf{y})R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\
&= \frac{1}{2t}\int_{\mathcal{M}} (\mathbf{x}-\mathbf{y})\cdot\nabla u(\mathbf{y})R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}.
\end{aligned}
$$

Here, Φ^i, $i=1,\cdots,d$ is the ith component of the parameterization function Φ and the parameterization function $\Phi = \Phi_{J(\mathbf{y})}$, $J(\mathbf{y})$ is the index function given in (5.2). In the rest of the proof, without introducing any confusion, we always drop the subscript of the parameterization function.

First, we split the residual $r(\mathbf{x})$ to four terms

$$
r(\mathbf{x}) = r_1(\mathbf{x}) + r_2(\mathbf{x}) + r_3(\mathbf{x}) - r_4(\mathbf{x})
$$

where

$$
\begin{aligned}
r_1(\mathbf{x}) &= \frac{1}{t}\int_{\mathcal{M}} \left(u(\mathbf{x})-u(\mathbf{y})-(\mathbf{x}-\mathbf{y})\cdot\nabla u(\mathbf{y})-\frac{1}{2}\eta^i\eta^j(\nabla^i\nabla^j u(\mathbf{y}))\right)R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}, \\
r_2(\mathbf{x}) &= \frac{1}{2t}\int_{\mathcal{M}} \eta^i\eta^j(\nabla^i\nabla^j u(\mathbf{y}))R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} - \int_{\mathcal{M}} \eta^i(\nabla^i\nabla^j u(\mathbf{y})\nabla^j\bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}, \\
r_3(\mathbf{x}) &= \int_{\mathcal{M}} \eta^i(\nabla^i\nabla^j u(\mathbf{y})\nabla^j\bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} + \int_{\mathcal{M}} \mathrm{div}\left(\eta^i(\nabla^i\nabla u(\mathbf{y}))\bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}, \\
r_4(\mathbf{x}) &= \int_{\mathcal{M}} \mathrm{div}\left(\eta^i(\nabla^i\nabla u(\mathbf{y}))\bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} + \int_{\mathcal{M}} \Delta_{\mathcal{M}}u(\mathbf{y})\bar{R}_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}.
\end{aligned}
$$

where ∇^i, $i = 1, \cdots, d$ is the ith component of the gradient ∇, η^i, $i = 1, \cdots, d$ is the ith component of $\eta(\mathbf{x}, \mathbf{y})$ defined in (5.3). To simplify the notation, we drop the variable $(\mathbf{x}, \mathbf{y})$ in the function $\eta(\mathbf{x}, \mathbf{y})$.

Next, we will prove the theorem by estimating above four terms one by one. First, we consider r_1. Let

$$d(\mathbf{x}, \mathbf{y}) = u(\mathbf{x}) - u(\mathbf{y}) - (\mathbf{x} - \mathbf{y}) \cdot \nabla u(\mathbf{y}) - \frac{1}{2}\eta^i\eta^j(\nabla^i\nabla^j u(\mathbf{y})).$$

we have

$$\begin{aligned}
\int_{\mathcal{M}} |r_1(\mathbf{x})|^2 \mathrm{d}\mu_{\mathbf{x}} &= \int_{\mathcal{M}} \left| \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) d(\mathbf{x}, \mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} \right|^2 \mathrm{d}\mu_{\mathbf{x}} \\
&\leq (\max_{\mathbf{y}})^2 \int_{\mathcal{M}} \left(\int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} \right) \left(\int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) |d(\mathbf{x}, \mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{y}} \right) \mathrm{d}\mu_{\mathbf{x}} \\
&\leq C \int_{\mathcal{M}} \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) |d(\mathbf{x}, \mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{x}}
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathcal{M}} \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) |d(\mathbf{x}, \mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{x}} &= \sum_{i=1}^{N} \int_{\mathcal{M}} \int_{\mathcal{O}_i} R_t(\mathbf{x}, \mathbf{y}) |d(\mathbf{x}, \mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{x}} \\
&= \sum_{i=1}^{N} \int_{\mathcal{O}_i} \left(\int_{\mathcal{M}_{\mathbf{y}}^t} R_t(\mathbf{x}, \mathbf{y}) |d(\mathbf{x}, \mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{x}} \right) \mathrm{d}\mu_{\mathbf{y}}.
\end{aligned}$$

Using Newton-Leibniz formula, we get

$$\begin{aligned}
d(\mathbf{x}, \mathbf{y}) &= u(\mathbf{x}) - u(\mathbf{y}) - (\mathbf{x} - \mathbf{y}) \cdot \nabla u(\mathbf{y}) - \frac{1}{2}\eta^i\eta^j(\nabla^i\nabla^j u(\mathbf{y})) \\
&= \xi^i \xi^{i'} \int_0^1 \int_0^1 \int_0^1 s_1 \frac{d}{ds_3}\left(\partial_i \Phi^j(\alpha + s_3 s_1 \xi) \partial_{i'} \Phi^{j'}(\alpha + s_3 s_2 s_1 \xi) \nabla^{j'} \nabla^j u(\Phi(\alpha + s_3 s_2 s_1 \xi)) \right) \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1 \\
&= \xi^i \xi^{i'} \xi^{i''} \int_0^1 \int_0^1 \int_0^1 s_1^2 s_2 \partial_i \Phi^j(\alpha + s_3 s_1 \xi) \partial_{i''}\partial_{i'} \Phi^{j'}(\alpha + s_3 s_2 s_1 \xi) \nabla^{j'} \nabla^j u(\Phi(\alpha + s_3 s_2 s_1 \xi)) \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1 \\
&\quad + \xi^i \xi^{i'} \xi^{i''} \int_0^1 \int_0^1 \int_0^1 s_1^2 \partial_{i''}\partial_i \Phi^j(\alpha + s_3 s_1 \xi) \partial_{i'} \Phi^{j'}(\alpha + s_3 s_2 s_1 \xi) \nabla^{j'} \nabla^j u(\Phi(\alpha + s_3 s_2 s_1 \xi)) \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1 \\
&\quad + \xi^i \xi^{i'} \xi^{i''} \int_0^1 \int_0^1 \int_0^1 s_1^2 s_2 \partial_i \Phi^j(\alpha + s_3 s_2 s_1 \xi) \partial_{i'} \Phi^{j'}(\alpha + s_3 s_2 s_1 \xi) \partial_{i''} \Phi^{j''}(\alpha + s_3 s_2 s_1 \xi) \\
&\qquad \nabla^{j''}\nabla^{j'}\nabla^j u(\Phi(\alpha + s_3 s_2 s_1 \xi)) \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1
\end{aligned}$$

Here, $\alpha = \alpha(\mathbf{x}, \mathbf{y}) = \Phi_{J(\mathbf{y})}^{-1}(\mathbf{y})$, $\xi = \xi(\mathbf{x}, \mathbf{y}) = \Phi_{J(\mathbf{y})}^{-1}(\mathbf{x}) - \Phi_{J(\mathbf{y})}^{-1}(\mathbf{y})$. In above derivation, we need the convexity property of the parameterization function to make sure all the integrals are well defined.

Using above equality and the smoothness of the parameterization functions, it is easy to show that

$$\begin{aligned}
&\int_{\mathcal{O}_i} \left(\int_{\mathcal{M}_{\mathbf{y}}^t} R_t(\mathbf{x}, \mathbf{y}) |d(\mathbf{x}, \mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{x}} \right) \mathrm{d}\mu_{\mathbf{y}} \\
&\leq Ct^3 \int_0^1 \int_0^1 \int_0^1 \int_{\mathcal{O}_i} \int_{\mathcal{M}_{\mathbf{y}}^t} R_t(\mathbf{x}, \mathbf{y}) \left| D^{2,3} u(\Phi_{J(\mathbf{y})}(\alpha + s_3 s_2 s_1 \xi)) \right|^2 \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\mu_{\mathbf{y}} \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1 \\
&\leq Ct^3 \max_{0 \leq s \leq 1} \int_{\mathcal{O}_i} \int_{\mathcal{M}_{\mathbf{y}}^t} R_t(\mathbf{x}, \mathbf{y}) \left| D^{2,3} u(\Phi_i(\alpha + s\xi)) \right|^2 \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\mu_{\mathbf{y}},
\end{aligned}$$

where we use the fact that $J(\mathbf{y}) = i$, $\mathbf{y} \in \mathcal{O}_i$ and

$$\left|D^{2,3}u(\mathbf{x})\right|^2 = \sum_{j,j',j''=1}^{d} |\nabla^{j''}\nabla^{j'}\nabla^j u(\mathbf{x})|^2 + \sum_{j,j'=1}^{d} |\nabla^{j'}\nabla^j u(\mathbf{x})|^2.$$

Let $\mathbf{z}_i = \Phi_i(\alpha + s\xi)$, $0 \leq s \leq 1$, then for any $\mathbf{y} \in \mathcal{O}_i \subset B_{\mathbf{q}_i}^{\delta}$ and $\mathbf{x} \in \mathcal{M}_{\mathbf{y}}^t$,

$$|\mathbf{z}_i - \mathbf{y}| \leq 2s|\xi| \leq 4s|\mathbf{x} - \mathbf{y}| \leq 8s\sqrt{t}, \quad |\mathbf{z}_i - \mathbf{q}_i| \leq |\mathbf{z}_i - \mathbf{y}| + |\mathbf{y} - \mathbf{q}_i| \leq \delta + 8s\sqrt{t}.$$

We can assume that t is small enough such that $8\sqrt{t} \leq \delta$, then we have

$$\mathbf{z}_i \in B_{\mathbf{q}_i}^{2\delta}.$$

After changing of variable, we obtain

$$\int_{\mathcal{O}_i} \int_{\mathcal{M}_{\mathbf{y}}^t} R_t(\mathbf{x},\mathbf{y}) \left|D^{2,3}u(\Phi_i(\alpha+s\xi))\right|^2 \mathrm{d}\mu_{\mathbf{x}}\mathrm{d}\mu_{\mathbf{y}}$$

$$\leq \frac{C}{\delta_0} \int_{\mathcal{O}_i} \int_{B_{\mathbf{q}_i}^{2\delta}} \frac{1}{s^k} R\left(\frac{|\mathbf{z}_i - \mathbf{y}|^2}{128s^2t}\right) \left|D^{2,3}u(\mathbf{z}_i)\right|^2 \mathrm{d}\mu_{\mathbf{z}_i}\mathrm{d}\mu_{\mathbf{y}}$$

$$= \frac{C}{\delta_0} \int_{\mathcal{O}_i} \frac{1}{s^k} R\left(\frac{|\mathbf{z}_i - \mathbf{y}|^2}{128s^2t}\right) \mathrm{d}\mu_{\mathbf{y}} \int_{B_{\mathbf{q}_i}^{2\delta}} \left|D^{2,3}u(\mathbf{z}_i)\right|^2 \mathrm{d}\mu_{\mathbf{z}_i}$$

$$\leq C \int_{B_{\mathbf{q}_i}^{2\delta}} \left|D^{2,3}u(\mathbf{x})\right|^2 \mathrm{d}\mu_{\mathbf{x}}.$$

This estimate would give us that

$$\|r_1(\mathbf{x})\|_{L^2(\mathcal{M})} \leq Ct^{1/2}\|u\|_{H^3(\mathcal{M})} \tag{5.5}$$

Now, we turn to estimate the gradient of r_1.

$$\int_{\mathcal{M}} |\nabla_{\mathbf{x}} r_1(\mathbf{x})|^2 \mathrm{d}\mu_{\mathbf{x}} \leq C \int_{\mathcal{M}} \left|\int_{\mathcal{M}} \nabla_{\mathbf{x}} R_t(\mathbf{x},\mathbf{y}) d(\mathbf{x},\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}}\right|^2 \mathrm{d}\mu_{\mathbf{x}}$$

$$+ C \int_{\mathcal{M}} \left|\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y}) \nabla_{\mathbf{x}} d(\mathbf{x},\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}}\right|^2 \mathrm{d}\mu_{\mathbf{x}}.$$

where $\nabla_{\mathbf{x}}$ is the gradient in $\mathcal{M}$ with respect to $\mathbf{x}$.

Using the same techniques in the calculation of $\|r_1(\mathbf{x})\|_{L^2(\mathcal{M})}$, we get that the first term of right hand side can bounded as follows

$$\int_{\mathcal{M}} \left|\int_{\mathcal{M}} \nabla_{\mathbf{x}} R_t(\mathbf{x},\mathbf{y}) d(\mathbf{x},\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}}\right|^2 \mathrm{d}\mu_{\mathbf{x}} \leq C\|u\|_{H^3(\mathcal{M})}^2.$$

The estimation of second term is a little involved. First, we have

$$\int_{\mathcal{M}} \left|\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y}) \nabla_{\mathbf{x}} d(\mathbf{x},\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}}\right|^2 \mathrm{d}\mu_{\mathbf{x}} \leq C \int_{\mathcal{M}} \left(\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})|\nabla_{\mathbf{x}} d(\mathbf{x},\mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{y}}\right) \mathrm{d}\mu_{\mathbf{x}}$$

$$= C \sum_{i=1}^{N} \int_{\mathcal{O}_i} \left(\int_{\mathcal{M}_{\mathbf{y}}^t} R_t(\mathbf{x},\mathbf{y})|\nabla_{\mathbf{x}} d(\mathbf{x},\mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{x}}\right) \mathrm{d}\mu_{\mathbf{y}}.$$

11

Also using Newton-Leibniz formula, we have

$$d(\mathbf{x},\mathbf{y}) = \xi^i\xi^{i'}\int_0^1\int_0^1 s_1\left(\partial_i\Phi^j(\alpha+s_1\xi)\partial_{i'}\Phi^{j'}(\alpha+s_2s_1\xi)\nabla^{j'}\nabla^j u(\Phi(\alpha+s_2s_1\xi))\right)\mathrm{d}s_2\mathrm{d}s_1$$

$$-\xi^i\xi^{i'}\int_0^1\int_0^1 s_1\left(\partial_i\Phi^j(\alpha)\partial_{i'}\Phi^{j'}(\alpha)\nabla^{j'}\nabla^j u(\Phi(\alpha))\right)\mathrm{d}s_2\mathrm{d}s_1$$

Then the gradient of $d(\mathbf{x},\mathbf{y})$ has following representation,

$$\nabla_{\mathbf{x}}d(\mathbf{x},\mathbf{y}) = \xi^i\xi^{i'}\nabla_{\mathbf{x}}\left(\int_0^1\int_0^1 s_1\left(\partial_i\Phi^j(\alpha+s_1\xi)\partial_{i'}\Phi^{j'}(\alpha+s_2s_1\xi)\nabla^{j'}\nabla^j u(\Phi(\alpha+s_2s_1\xi))\right)\mathrm{d}s_2\mathrm{d}s_1\right)$$

$$+\nabla_{\mathbf{x}}\left(\xi^i\xi^{i'}\right)\int_0^1\int_0^1\int_0^1 s_1\frac{d}{ds_3}\left(\partial_i\Phi^j(\alpha+s_3s_1\xi)\partial_{i'}\Phi^{j'}(\alpha+s_3s_2s_1\xi)\nabla^{j'}\nabla^j u(\Phi(\alpha+s_3s_2s_1\xi))\right)\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1$$

$$= d_1(\mathbf{x},\mathbf{y}) + d_2(\mathbf{x},\mathbf{y}).$$

For d_1, we have

$$\int_{\mathcal{O}_i}\left(\int_{\mathcal{M}^t_{\mathbf{y}}}R_t(\mathbf{x},\mathbf{y})|d_1(\mathbf{x},\mathbf{y})|^2\mathrm{d}\mu_{\mathbf{x}}\right)\mathrm{d}\mu_{\mathbf{y}} \leq Ct^2\max_{0\leq s\leq1}\int_{\mathcal{O}_i}\left(\int_{\mathcal{M}^t_{\mathbf{y}}}R_t(\mathbf{x},\mathbf{y})|D^{2,3}u(\Phi_i(\alpha+s\xi))|^2\mathrm{d}\mu_{\mathbf{x}}\right)\mathrm{d}\mu_{\mathbf{y}},$$

which means that

$$\int_{\mathcal{O}_i}\left(\int_{\mathcal{M}^t_{\mathbf{y}}}R_t(\mathbf{x},\mathbf{y})|d_1(\mathbf{x},\mathbf{y})|^2\mathrm{d}\mu_{\mathbf{x}}\right)\mathrm{d}\mu_{\mathbf{y}} \leq C\int_{B^{2\delta}_{\mathbf{q}_i}}|D^{2,3}u(\mathbf{x})|^2\mathrm{d}\mu_{\mathbf{x}} \tag{5.6}$$

For d_2, we have

$$d_2(\mathbf{x},\mathbf{y})$$
$$= \nabla_{\mathbf{x}}\left(\xi^i\xi^{i'}\right)\int_{[0,1]^3}s_1\frac{d}{ds_3}\left(\partial_i\Phi^j(\alpha+s_3s_1\xi)\partial_{i'}\Phi^{j'}(\alpha+s_3s_2s_1\xi)\nabla^{j'}\nabla^j u(\Phi(\alpha+s_3s_2s_1\xi))\right)\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1$$

$$= \nabla_{\mathbf{x}}\left(\xi^i\xi^{i'}\right)\xi^{i''}\int_{[0,1]^3}s_1^2 s_2\partial_i\Phi^j(\alpha+s_3s_1\xi)\partial_{i''}\partial_{i'}\Phi^{j'}(\alpha+s_3s_2s_1\xi)\nabla^{j'}\nabla^j u(\Phi(\alpha+s_3s_2s_1\xi))\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1$$

$$+\nabla_{\mathbf{x}}\left(\xi^i\xi^{i'}\right)\xi^{i''}\int_{[0,1]^3}s_1^2\partial_{i''}\partial_i\Phi^j(\alpha+s_3s_1\xi)\partial_{i'}\Phi^{j'}(\alpha+s_3s_2s_1\xi)\nabla^{j'}\nabla^j u(\Phi(\alpha+s_3s_2s_1\xi))\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1$$

$$+\nabla_{\mathbf{x}}\left(\xi^i\xi^{i'}\right)\xi^{i''}\int_{[0,1]^3}s_1^2 s_2\partial_i\Phi^j(\alpha+s_2s_1\xi)\partial_{i'}\Phi^{j'}(\alpha+s_3s_2s_1\xi)\partial_{i''}\Phi^{j''}(\alpha+s_3s_2s_1\xi)$$

$$\nabla^{j''}\nabla^{j'}\nabla^j u(\Phi(\alpha+s_3s_2s_1\xi))\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1$$

This formula tells us that

$$\int_{\mathcal{O}_i}\left(\int_{\mathcal{M}^t_{\mathbf{y}}}R_t(\mathbf{x},\mathbf{y})|d_2(\mathbf{x},\mathbf{y})|^2\mathrm{d}\mu_{\mathbf{x}}\right)\mathrm{d}\mu_{\mathbf{y}} \leq Ct^2\max_{0\leq s\leq1}\int_{\mathcal{O}_i}\left(\int_{\mathcal{M}^t_{\mathbf{y}}}R_t(\mathbf{x},\mathbf{y})|D^{2,3}u(\Phi(\alpha+s\xi))|^2\mathrm{d}\mu_{\mathbf{x}}\right)\mathrm{d}\mu_{\mathbf{y}}.$$

Using the same arguments as that in the calculation of $\|r_1\|_{L^2(\mathcal{M})}$, we have

$$\int_{\mathcal{O}_i}\left(\int_{\mathcal{M}^t_{\mathbf{y}}}R_t(\mathbf{x},\mathbf{y})|d_2(\mathbf{x},\mathbf{y})|^2\mathrm{d}\mu_{\mathbf{x}}\right)\mathrm{d}\mu_{\mathbf{y}} \leq C\int_{B^{2\delta}_{\mathbf{q}_i}}|D^3 u(\mathbf{x})|^2\mathrm{d}\mu_{\mathbf{x}} \tag{5.7}$$

Combining (5.6) and (5.7), we have

$$\|\nabla r_1(\mathbf{x})\|_{L^2(\mathcal{M})} \leq C\|u\|_{H^3(\mathcal{M})} \tag{5.8}$$

12

For r_2, first, notice that

$$\nabla^j \bar{R}_t(\mathbf{x}, \mathbf{y}) = \frac{1}{2t} \partial_{m'} \Phi^j(\alpha) g^{m'n'} \partial_{n'} \Phi^i(\alpha)(x^i - y^i) R_t(\mathbf{x}, \mathbf{y}),$$

$$\frac{\eta^j}{2t} R_t(\mathbf{x}, \mathbf{y}) = \frac{1}{2t} \partial_{m'} \Phi^j(\alpha) g^{m'n'} \partial_{n'} \Phi^i(\alpha) \xi^{i'} \partial_{i'} \Phi^i R_t(\mathbf{x}, \mathbf{y}).$$

Then, we have

$$\nabla^j \bar{R}_t(\mathbf{x}, \mathbf{y}) - \frac{\eta^j}{2t} R_t(\mathbf{x}, \mathbf{y})$$

$$= \frac{1}{2t} \partial_{m'} \Phi^i g^{m'n'} \partial_{n'} \Phi^j \left(x^j - y^j - \xi^{i'} \partial_{i'} \Phi^j \right) R_t(\mathbf{x}, \mathbf{y})$$

$$= \frac{1}{2t} \xi^{i'} \xi^{j'} \partial_{m'} \Phi^i g^{m'n'} \partial_{n'} \Phi^j \left(\int_0^1 \int_0^1 s \partial_{j'} \partial_{i'} \Phi^j(\alpha + \tau s \xi) \mathrm{d}\tau \mathrm{d}s \right) R_t(\mathbf{x}, \mathbf{y})$$

Thus, we get

$$\left| \nabla^j \bar{R}_t(\mathbf{x}, \mathbf{y}) - \frac{\eta^j}{2t} R_t(\mathbf{x}, \mathbf{y}) \right| \leq \frac{C|\xi|^2}{t} R_t(\mathbf{x}, \mathbf{y})$$

$$\left| \nabla_{\mathbf{x}} \left(\nabla^j \bar{R}_t(\mathbf{x}, \mathbf{y}) - \frac{\eta^j}{2t} R_t(\mathbf{x}, \mathbf{y}) \right) \right| \leq \frac{C|\xi|}{t} R_t(\mathbf{x}, \mathbf{y}) + \frac{C|\xi|^3}{t^2} |R_t'(\mathbf{x}, \mathbf{y})|$$

Then, we have following bound for r_2,

$$\int_{\mathcal{M}} |r_2(\mathbf{x})|^2 \mathrm{d}\mu_{\mathbf{x}} \tag{5.9}$$

$$\leq Ct \int_{\mathcal{M}} \left(\int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) |D^2 u(\mathbf{y})| \mathrm{d}\mu_{\mathbf{y}} \right)^2 \mathrm{d}\mu_{\mathbf{x}}$$

$$\leq Ct \int_{\mathcal{M}} \left(\int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} \right) \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) |D^2 u(\mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{x}}$$

$$\leq Ct \max_{\mathbf{y}} \left(\int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) \mathrm{d}\mu_{\mathbf{x}} \right) \int_{\mathcal{M}} |D^2 u(\mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{y}}$$

$$\leq Ct \|u\|_{H^2(\mathcal{M})}^2.$$

Similarly, we have

$$\int_M |\nabla r_2(\mathbf{x})|^2 \mathrm{d}\mu_{\mathbf{x}} \tag{5.10}$$

$$\leq Ct \int_{\mathcal{M}} \left(\int_{\mathcal{M}} \nabla_{\mathbf{x}} R_t(\mathbf{x}, \mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} \right) \int_{\mathcal{M}} \nabla_{\mathbf{x}} R_t(\mathbf{x}, \mathbf{y}) |D^2 u(\mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{x}}$$

$$\leq C\sqrt{t} \max_{\mathbf{y}} \left(\int_{\mathcal{M}} \nabla_{\mathbf{x}} R_t(\mathbf{x}, \mathbf{y}) \mathrm{d}\mu_{\mathbf{x}} \right) \int_{\mathcal{M}} |D^2 u(\mathbf{y})|^2 \mathrm{d}\mu_{\mathbf{y}}$$

$$\leq C \|u\|_{H^2(\mathcal{M})}^2.$$

r_3 is relatively easy to estimate by using the well known Gauss formula.

$$r_3(\mathbf{x}) = \int_{\partial \mathcal{M}} n^j \eta^i (\nabla^i \nabla^j u(\mathbf{y})) \bar{R}_t(\mathbf{x}, \mathbf{y}) \mathrm{d}\tau_{\mathbf{y}} - \int_{\mathcal{M}} \eta^i (\nabla^i \nabla^j u(\mathbf{y})) \bar{R}_t(\mathbf{x}, \mathbf{y}) \nabla^j \mathrm{d}\mu_{\mathbf{y}}$$

$$= \tilde{I}_{bd} - \int_{\mathcal{M}} \eta^i (\nabla^i \nabla^j u(\mathbf{y})) \bar{R}_t(\mathbf{x}, \mathbf{y}) \nabla^j \mathrm{d}\mu_{\mathbf{y}}$$

13

where $\tilde{I}_{bd} = \int_{\partial\mathcal{M}} n^j \eta^i (\nabla^i \nabla^j u(\mathbf{y})) \bar{R}_t(\mathbf{x}, \mathbf{y}) \mathrm{d}\tau_\mathbf{y}$.

Using the assumption that $p \in C^1(\mathcal{M})$, it is easy to get that

$$\|r_3 - \tilde{I}_{bd}\|_{L^2(\mathcal{M})} \leq C\sqrt{t}\|u\|_{H^2(\mathcal{M})}, \tag{5.11}$$

$$\|\nabla(r_3 - \tilde{I}_{bd})\|_{L^2(\mathcal{M})} \leq C\|u\|_{H^2(\mathcal{M})}. \tag{5.12}$$

Now, we turn to bound the last term r_4. Notice that

$$\begin{aligned}
\nabla^j \left(\nabla^j u(\mathbf{y})\right) &= (\partial_{k'}\Phi^j)g^{k'l'}\partial_{l'}\left((\partial_{m'}\Phi^j)g^{m'n'}(\partial_{n'}u)\right) \\
&= (\partial_{k'}\Phi^j)g^{k'l'}\left(\partial_{l'}(\partial_{m'}\Phi^j)\right)g^{m'n'}(\partial_{n'}u) \\
&\quad + (\partial_{k'}\Phi^j)g^{k'l'}(\partial_{m'}\Phi^j)\partial_{l'}\left(g^{m'n'}(\partial_{n'}u)\right) \\
&= \frac{1}{\sqrt{\det G}}(\partial_{m'}\sqrt{\det G})g^{m'n'}(\partial_{n'}u) + \partial_{m'}\left(g^{m'n'}(\partial_{n'}u)\right) \\
&= \frac{1}{\sqrt{\det G}}\partial_{m'}\left(\sqrt{\det G}g^{m'n'}(\partial_{n'}u)\right) = \Delta_\mathcal{M}u(\mathbf{y}). \tag{5.13}
\end{aligned}$$

where $\det G$ is the determinant of G and $G = (g_{ij})_{i,j=1,\cdots,k}$. Here we use the fact that

$$\begin{aligned}
(\partial_{k'}\Phi^j)g^{k'l'}\left(\partial_{l'}(\partial_{m'}\Phi^j)\right) &= (\partial_{k'}\Phi^j)g^{k'l'}\left(\partial_{m'}(\partial_{l'}\Phi^j)\right) \\
&= (\partial_{m'}(\partial_{k'}\Phi^j))g^{k'l'}(\partial_{l'}\Phi^j) \\
&= \frac{1}{2}g^{k'l'}\partial_{m'}(g_{k'l'}) \\
&= \frac{1}{\sqrt{\det G}}(\partial_{m'}\sqrt{\det G}).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
g^{i'j'}(\partial_{j'}\Phi^j)(\partial_{i'}\xi^l)(\partial_l\Phi^i)(\nabla^i\nabla^j u(\mathbf{y})) & \tag{5.14} \\
= -g^{i'j'}(\partial_{j'}\Phi^j)(\partial_{i'}\Phi^i)(\nabla^i\nabla^j u(\mathbf{y})) & \\
= -g^{i'j'}(\partial_{j'}\Phi^j)(\partial_{i'}\Phi^i)(\partial_{m'}\Phi^i)g^{m'n'}\partial_{n'}\left(\nabla^j u(\mathbf{y})\right) & \\
= -g^{i'j'}(\partial_{j'}\Phi^j)\partial_{i'}\left(\nabla^j u(\mathbf{y})\right) & \\
= -\nabla^j\left(\nabla^j u(\mathbf{y})\right). &
\end{aligned}$$

where the first equalities are due to that $\partial_{i'}\xi^l = -\delta^l_{i'}$. Then we have

$$\begin{aligned}
&\mathrm{div}\left(\eta^i(\nabla^i\nabla^j u(\mathbf{y}))\right) + \Delta_\mathcal{M}u(\mathbf{y}) \\
&= \frac{1}{\sqrt{\det G}}\partial_{i'}\left(\sqrt{\det G}\,g^{i'j'}(\partial_{j'}\Phi^j)\xi^l(\partial_l\Phi^i)(\nabla^i\nabla^j u(\mathbf{y}))\right) - g^{i'j'}(\partial_{j'}\Phi^j)(\partial_{i'}\xi^l)(\partial_l\Phi^i)(\nabla^i\nabla^j u(\mathbf{y})) \\
&= \frac{\xi^l}{\sqrt{\det G}}\partial_{i'}\left(\sqrt{\det G}\,g^{i'j'}(\partial_{j'}\Phi^j)(\partial_l\Phi^i)(\nabla^i\nabla^j u(\mathbf{y}))\right).
\end{aligned}$$

Here we use the equalities (5.13), (5.14), $\eta^i = \xi^l \partial_{i'}\Phi^l$ and the definition of div,

$$\mathrm{div}X = \frac{1}{\sqrt{\det G}}\partial_{i'}(\sqrt{\det G}\,g^{i'j'}\partial_{j'}\Phi^k X^k). \tag{5.15}$$

where X is a smooth tangent vector field on $\mathcal{M}$ and $(X^1, \ldots, X^d)^t$ is its representation in embedding coordinates.

14

Hence,

$$r_4(\mathbf{x}) = \int_{\mathcal{M}} \frac{\xi^l}{\sqrt{\det G}} \, \partial_{i'} \left(\sqrt{\det G} \, g^{i'j'} (\partial_{j'} \Phi^j)(\partial_l \Phi^i)(\nabla^i \nabla^j u(\mathbf{y})) \right) \bar{R}_t(\mathbf{x}, \mathbf{y}) \mathrm{d}\mu_{\mathbf{y}}$$

Then it is easy to get that

$$\begin{aligned}
\|r_4(\mathbf{x})\|_{L^2(\mathcal{M})} &\leq Ct^{1/2} \|u\|_{H^3(\mathcal{M})}, & (5.16) \\
\|\nabla r_4(\mathbf{x})\|_{L^2(\mathcal{M})} &\leq C \|u\|_{H^3(\mathcal{M})}. & (5.17)
\end{aligned}$$

By combining (5.5),(5.8),(5.9),(5.10),(5.11),(5.12),(5.16),(5.17), we know that

$$\begin{aligned}
\|r - \tilde{I}_{bd}\|_{L^2(\mathcal{M})} &\leq Ct^{1/2} \|u\|_{H^3(\mathcal{M})}, & (5.18) \\
\|\nabla(r - \tilde{I}_{bd})\|_{L^2(\mathcal{M})} &\leq C \|u\|_{H^3(\mathcal{M})}. & (5.19)
\end{aligned}$$

Using the definition of I_{bd} and $\tilde{I}_{bd}$, we obtain

$$I_{bd} - \tilde{I}_{bd} = \int_{\partial \mathcal{M}} n^j(\mathbf{y})(\mathbf{x} - \mathbf{y} - \eta(\mathbf{x}, \mathbf{y})) \cdot (\nabla \nabla^j u(\mathbf{y})) \bar{R}_t(\mathbf{x}, \mathbf{y}) \mathrm{d}\tau_{\mathbf{y}}$$

Using the definition of $\eta(\mathbf{x}, \mathbf{y})$, it is easy to check that

$$|\mathbf{x} - \mathbf{y} - \eta(\mathbf{x}, \mathbf{y})| = O(|\mathbf{x} - \mathbf{y}|^2), \quad |\nabla_{\mathbf{x}}(\mathbf{x} - \mathbf{y} - \eta(\mathbf{x}, \mathbf{y}))| = O(|\mathbf{x} - \mathbf{y}|)$$

which implies that

$$\begin{aligned}
\|I_{bd} - \tilde{I}_{bd}\|_{L^2(\mathcal{M})} &\leq Ct^{3/4} \|u\|_{H^2(\mathcal{M})}, & (5.20) \\
\|\nabla(I_{bd} - \tilde{I}_{bd})\|_{L^2(\mathcal{M})} &\leq Ct^{1/4} \|u\|_{H^3(\mathcal{M})}. & (5.21)
\end{aligned}$$

The theorem is proved by putting (5.18), (5.19), (5.20), (5.21) together. $\qquad \square$

6   Error analysis of the discretization (Theorem 4.2)

In this section, we estimate the discretization error introduced by approximating the integrals in (4.2) that is to prove Theorem 4.2. To simplify the notation, we introduce a intermediate operator defined as follows,

$$L_{t,h} u(\mathbf{x}) = \frac{1}{t} \sum_{\mathbf{p}_j \in P} R_t(\mathbf{x}, \mathbf{p}_j)(u(\mathbf{x}) - u(\mathbf{p}_j)) V_j. \tag{6.1}$$

If $u_{t,h} = I_{\mathbf{f}}(\mathbf{u})$ with $\mathbf{u}$ satisfying Equation (3.2), one can verify that the following equation is satisfied,

$$L_{t,h} u_{t,h}(\mathbf{x}) = \sum_{\mathbf{p}_j \in P} \bar{R}_t(\mathbf{x}, \mathbf{p}_j) f(\mathbf{p}_j) V_j. \tag{6.2}$$

We introduce a discrete operator $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}^n$ where $n = |P|$. For any $\mathbf{u} = (u_1, \cdots, u_n)^t$, denote

$$(\mathcal{L}\mathbf{u})_i = \frac{1}{t} \sum_{\mathbf{p}_j \in P} R_t(\mathbf{p}_i, \mathbf{p}_j)(u_i - u_j) V_j. \tag{6.3}$$

For this operator, we have the following important theorem.

Theorem 6.1. *Under the assumptions in Assumption (3.1), there exist constants $C > 0$, $C_0 > 0$ independent on t so that for any $\mathbf{u} = (u_1, \cdots, u_n)^t \in \mathbb{R}^d$ with $\sum_{i=1}^n u_i V_i = 0$ and for any sufficient small t and $\frac{h(P, \mathbf{V}, \mathcal{M})}{\sqrt{t}}$*

$$\langle \mathbf{u}, \mathcal{L}\mathbf{u} \rangle_{\mathbf{V}} \geq C(1 - \frac{C_0 h(P, \mathbf{V}, \mathcal{M})}{\sqrt{t}}) \langle \mathbf{u}, \mathbf{u} \rangle_{\mathbf{V}} \tag{6.4}$$

where $\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{V}} = \sum_{i=1}^n u_i v_i V_i$ for any $\mathbf{u} = (u_1, \cdots, u_n)$, $\mathbf{v} = (v_1, \cdots, v_n)$.

The proof of the above theorem is deferred to Appendix D.

It has an easy corollary which gives a priori estimate of $\mathbf{u} = (u_1, \cdots, u_n)^t$ solving the discrete problem (3.2).

Lemma 6.1. *Suppose $\mathbf{u} = (u_1, \cdots, u_n)^t$ with $\sum_i u_i V_i = 0$ solves the problem (3.2) and $\mathbf{f} = (f(\mathbf{p}_1), \cdots, f(\mathbf{p}_n))^t$ for $f \in C(\mathcal{M})$, there exists a constant $C > 0$ such that*

$$\left(\sum_{i=1}^n u_i^2 V_i \right)^{1/2} \leq C\|f\|_{\infty}, \tag{6.5}$$

provided t and $\frac{h(P, \mathbf{V}, \mathcal{M})}{\sqrt{t}}$ are small enough.

Proof. From Theorem 6.1, we have

$$\sum_{i=1}^n u_i^2 V_i \leq \sum_{i=1}^n \left(\sum_{\mathbf{p}_j \in P} \bar{R}_t(\mathbf{p}_i, \mathbf{p}_j) f_j V_j \right) u_i V_i$$

$$\leq \left(\sum_{i=1}^n u_i^2 V_i \right)^{1/2} \left(\sum_{i=1}^n \left(\|f\|_{\infty} \sum_{p_j \in P} \bar{R}_t(\mathbf{p}_i, \mathbf{p}_j) V_j \right)^2 V_i \right)^{1/2}$$

$$\leq C \left(\sum_{i=1}^n u_i^2 V_i \right)^{1/2} \|f\|_{\infty}.$$

This proves the lemma. $\qquad \square$

We are now ready to prove Theorem 4.2.

Proof. of Theorem 4.2
Denote

$$u_{t,h}(\mathbf{x}) = \frac{1}{w_{t,h}(\mathbf{x})} \left(\sum_{\mathbf{p}_j \in P} R_t(\mathbf{x}, \mathbf{p}_j) u_j V_j - t \sum_{\mathbf{p}_j \in P} \bar{R}_t(\mathbf{x}, \mathbf{p}_j) f_j V_j \right) \tag{6.6}$$

where $\mathbf{u} = (u_1, \cdots, u_n)^t$ with $\sum_{i=1}^n u_i V_i = 0$ solves the problem (3.2), $f_j = f(\mathbf{p}_j)$ and $w_{t,h}(\mathbf{x}) = \sum_{\mathbf{p}_j \in P} R_t(\mathbf{x}, \mathbf{p}_j) V_j$. For convenience, we set

$$a_{t,h}(\mathbf{x}) = \frac{1}{w_{t,h}(\mathbf{x})} \sum_{\mathbf{p}_j \in P} R_t(\mathbf{x}, \mathbf{p}_j) u_j V_j,$$

$$c_{t,h}(\mathbf{x}) = -\frac{t}{w_{t,h}(\mathbf{x})} \sum_{\mathbf{p}_j \in P} \bar{R}_t(\mathbf{x}, \mathbf{p}_j) f(\mathbf{p}_j) V_j,$$

and thus $u_{t,h} = a_{t,h} + c_{t,h}$.

In the proof, to simplify the notation, we denote $h = h(P, \mathbf{V}, \mathcal{M})$ and $n = |P|$.
First we upper bound $\|L_t(u_{t,h}) - L_{t,h}(u_{t,h})\|_{L^2(\mathcal{M})}$. For $c_{t,h}$, we have

$$|(L_t c_{t,h} - L_{t,h} c_{t,h})(\mathbf{x})|$$

$$= \frac{1}{t} \left| \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y})(c_{t,h}(\mathbf{x}) - c_{t,h}(\mathbf{y})) \mathrm{d}\mu_{\mathbf{y}} - \sum_{\mathbf{p}_j \in P} R_t(\mathbf{x}, \mathbf{p}_j)(c_{t,h}(\mathbf{x}) - c_{t,h}(\mathbf{p}_j)) V_j \right|$$

$$\leq \frac{1}{t} |c_{t,h}(\mathbf{x})| \left| \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} - \sum_{\mathbf{p}_j \in P} R_t(\mathbf{x}, \mathbf{p}_j) V_j \right|$$

$$+ \frac{1}{t} \left| \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) c_{t,h}(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} - \sum_{\mathbf{p}_j \in P} R_t(\mathbf{x}, \mathbf{p}_j) c_{t,h}(\mathbf{p}_j) V_j \right|$$

$$\leq \frac{Ch}{t^{3/2}} |c_{t,h}(\mathbf{x})| + \frac{Ch}{t^{3/2}} \|c_{t,h}\|_{C^1(\mathcal{M})}$$

$$\leq \frac{Ch}{t^{3/2}} t \|f\|_\infty + \frac{Ch}{t^{3/2}} (t\|f\|_\infty + t^{1/2}\|f\|_\infty) \leq \frac{Ch}{t} \|f\|_\infty.$$

For $a_{t,h}$, we have

$$\int_{\mathcal{M}} (a_{t,h}(\mathbf{x}))^2 \left| \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} - \sum_{\mathbf{p}_j \in P} R_t(\mathbf{x}, \mathbf{p}_j) V_j \right|^2 \mathrm{d}\mu_{\mathbf{x}} \tag{6.7}$$

$$\leq \frac{Ch^2}{t} \int_{\mathcal{M}} (a_{t,h}(\mathbf{x}))^2 \mathrm{d}\mu_{\mathbf{x}} \leq \frac{Ch^2}{t} \int_{\mathcal{M}} \left(\frac{1}{w_{t,h}(\mathbf{x})} \sum_{\mathbf{p}_j \in P} R_t(\mathbf{x}, \mathbf{p}_j) u_j V_j \right)^2 \mathrm{d}\mu_{\mathbf{x}}$$

$$\leq \frac{Ch^2}{t} \int_{\mathcal{M}} \left(\sum_{\mathbf{p}_j \in P} R_t(\mathbf{x}, \mathbf{p}_j) u_j^2 V_j \right) \left(\sum_{\mathbf{p}_j \in P} R_t(\mathbf{x}, \mathbf{p}_j) V_j \right) \mathrm{d}\mu_{\mathbf{x}}$$

$$\leq \frac{Ch^2}{t} \left(\sum_{j=1}^n u_j^2 V_j \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{p}_j) \mathrm{d}\mu_{\mathbf{x}} \right) \leq \frac{Ch^2}{t} \sum_{j=1}^n u_j^2 V_j.$$

Let

$$A = C_t \int_{\mathcal{M}} \frac{1}{w_{t,h}(\mathbf{y})} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t} \right) R\left(\frac{|\mathbf{p}_i - \mathbf{y}|^2}{4t} \right) \mathrm{d}\mu_{\mathbf{y}}$$

$$- C_t \sum_{\mathbf{p}_j \in P} \frac{1}{w_{t,h}(\mathbf{p}_j)} R\left(\frac{|\mathbf{x} - \mathbf{p}_j|^2}{4t} \right) R\left(\frac{|\mathbf{p}_i - \mathbf{p}_j|^2}{4t} \right) V_j.$$

We have $|A| < \frac{Ch}{t^{1/2}}$ for some constant C independent of t. In addition, notice that only when $|\mathbf{x} - \mathbf{p}_i|^2 \leq 16t$ is $A \neq 0$, which implies

$$|A| \leq \frac{1}{\delta_0} |A| R\left(\frac{|\mathbf{x} - \mathbf{p}_i|^2}{32t} \right).$$

17

Then we have

$$\int_{\mathcal{M}} \left| \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) a_{t,h}(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} - \sum_{\mathbf{p}_j \in P} R_t(\mathbf{x}, \mathbf{p}_j) a_{t,h}(\mathbf{p}_j) V_j \right|^2 \mathrm{d}\mu_{\mathbf{x}} \tag{6.8}$$

$$= \int_{\mathcal{M}} \left(\sum_{i=1}^{n} C_t u_i V_i A \right)^2 \mathrm{d}\mu_{\mathbf{x}} \le \frac{Ch^2}{t} \int_{\mathcal{M}} \left(\sum_{i=1}^{n} C_t |u_i| V_i R \left(\frac{|\mathbf{x} - \mathbf{p}_i|^2}{32t} \right) \right)^2 \mathrm{d}\mu_{\mathbf{x}}$$

$$\le \frac{Ch^2}{t} \int_{\mathcal{M}} \left(\sum_{i=1}^{n} C_t R \left(\frac{|\mathbf{x} - \mathbf{p}_i|^2}{32t} \right) u_i^2 V_i \right) \left(\sum_{\mathbf{p}_i \in P} C_t R \left(\frac{|\mathbf{x} - \mathbf{p}_i|^2}{32t} \right) V_i \right) \mathrm{d}\mu_{\mathbf{x}}$$

$$\le \frac{Ch^2}{t} \sum_{i=1}^{n} \left(\int_{\mathcal{M}} C_t R \left(\frac{|\mathbf{x} - \mathbf{p}_i|^2}{32t} \right) \mathrm{d}\mu_{\mathbf{x}} \left(u_i^2 V_i \right) \right) \le \frac{Ch^2}{t} \left(\sum_{i=1}^{n} u_i^2 V_i \right).$$

Combining Equation (6.7), (6.8) and Lemma 6.1,

$$\|L_t a_{t,h} - L_{t,h} a_{t,h}\|_{L^2(\mathcal{M})}$$

$$= \left(\int_{M} \left| (L_t(a_{t,h}) - L_{t,h}(a_{t,h})) (\mathbf{x}) \right|^2 \mathrm{d}\mu_{\mathbf{x}} \right)^{1/2}$$

$$\le \frac{1}{t} \left(\int_{\mathcal{M}} (a_{t,h}(\mathbf{x}))^2 \left| \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} - \sum_{\mathbf{p}_j \in P} R_t(\mathbf{x}, \mathbf{p}_j) V_j \right|^2 \mathrm{d}\mu_{\mathbf{x}} \right)^{1/2}$$

$$+ \frac{1}{t} \left(\int_{\mathcal{M}} \left| \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) a_{t,h}(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} - \sum_{\mathbf{p}_j \in P} R_t(\mathbf{x}, \mathbf{p}_j) a_{t,h}(\mathbf{p}_j) V_j \right|^2 \mathrm{d}\mu_{\mathbf{x}} \right)^{1/2}$$

$$\le \frac{Ch}{t^{3/2}} \left(\sum_{i=1}^{n} u_i^2 V_i \right)^{1/2} \le \frac{Ch}{t^{3/2}} \|f\|_{\infty}.$$

Assembling the parts together, we have the following upper bound.

$$\|L_t u_{t,h} - L_{t,h} u_{t,h}\|_{L^2(\mathcal{M})} \tag{6.9}$$
$$\le \|L_t a_{t,h} - L_{t,h} a_{t,h}\|_{L^2(\mathcal{M})} + \|L_t c_{t,h} - L_{t,h} c_{t,h}\|_{L^2(\mathcal{M})}$$
$$\le \frac{Ch}{t^{3/2}} \|f\|_{\infty} + \frac{Ch}{t} \|f\|_{\infty} \le \frac{Ch}{t^{3/2}} \|f\|_{\infty}$$

At the same time, since u_t respectively $u_{t,h}$ solves equation (4.2) respectively equation (6.2), we have

$$\|L_t(u_t) - L_{t,h}(u_{t,h})\|_{L^2(\mathcal{M})} \tag{6.10}$$

$$= \left(\int_{\mathcal{M}} \left((L_t u_t - L_{t,h} u_{t,h}) (\mathbf{x}) \right)^2 \mathrm{d}\mu_{\mathbf{x}} \right)^{1/2}$$

$$= \left(\int_{\mathcal{M}} \left(\int_{\mathcal{M}} \bar{R}_t(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) - \sum_{\mathbf{p}_j \in P} \bar{R}_t(\mathbf{x}, \mathbf{p}_j) f(\mathbf{p}_j) V_j \right)^2 \mathrm{d}\mu_{\mathbf{x}} \right)^{1/2}$$

$$\le \frac{Ch}{t^{1/2}} \|f\|_{C^1(\mathcal{M})}.$$

The complete L^2 estimate follows from Equation (6.9) and (6.10).

The estimate of the gradient, $\|\nabla(L_t(u_t) - L_{t,h}(u_{t,h}))\|_{L^2(\mathcal{M})}$, can be obtained similarly. The details can be found in Appendix E. $\qquad\square$

7 Stability analysis (Theorem 4.1 and 4.4)

To prove Theorem 4.1 and 4.4, we need following two theorems regarding the coercivity of the operator L_t.

Theorem 7.1. *For any function $u \in L^2(\mathcal{M})$, there exists a constant $C > 0$ independent on t and u, such that*

$$\langle u, L_t u \rangle_{\mathcal{M}} \geq C \int_{\mathcal{M}} |\nabla v|^2 \mathrm{d}\mu_{\mathbf{x}} \tag{7.1}$$

where $\langle f, g \rangle_{\mathcal{M}} = \int_{\mathcal{M}} f(\mathbf{x}) g(\mathbf{x}) \mathrm{d}\mu_{\mathbf{x}}$ for any $f, g \in L_2(\mathcal{M})$, and

$$v(\mathbf{x}) = \frac{C_t}{w_t(\mathbf{x})} \int_{\mathcal{M}} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t}\right) u(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}}, \tag{7.2}$$

and $w_t(\mathbf{x}) = C_t \int_{\mathcal{M}} R\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right) \mathrm{d}\mu_{\mathbf{y}}$.

Theorem 7.2. *Assume both $\mathcal{M}$ and $\partial\mathcal{M}$ are C^∞. There exists a constant $C > 0$ independent on t so that for any function $u \in L_2(\mathcal{M})$ with $\int_{\mathcal{M}} u(\mathbf{x}) \mathrm{d}\mu_{\mathbf{x}} = 0$ and for any sufficient small t*

$$\langle u, L_t u \rangle_{\mathcal{M}} \geq C \|u\|^2_{L_2(\mathcal{M})} \tag{7.3}$$

Theorem 4.1 is a direct corollary of following two lemmas.

Lemma 7.1. *For any function $u \in L^2(\mathcal{M})$, there exists a constant $C > 0$ independent on t and u, such that*

$$\frac{C_t}{t} \int_{\mathcal{M}} \int_{\mathcal{M}} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{32t}\right) (u(\mathbf{x}) - u(\mathbf{y}))^2 \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\mu_{\mathbf{y}} \geq C \int_{\mathcal{M}} |\nabla v|^2 \mathrm{d}\mu_{\mathbf{x}}$$

where v is the same as defined in (7.2).

Lemma 7.2. *If t is small enough, then for any function $u \in L^2(\mathcal{M})$, there exists a constant $C > 0$ independent on t and u, such that*

$$\int_{\mathcal{M}} \int_{\mathcal{M}} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{32t}\right) (u(\mathbf{x}) - u(\mathbf{y}))^2 \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\mu_{\mathbf{y}} \leq C \int_{\mathcal{M}} \int_{\mathcal{M}} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t}\right) (u(\mathbf{x}) - u(\mathbf{y}))^2 \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\mu_{\mathbf{y}}.$$

The proofs of the above two lemmas are put in Appendix B and C. Once we have Lemma 7.1 and Lemma 7.2, Theorem 7.1 becomes obvious by noticing that:

$$
\begin{aligned}
\langle u, L_t u \rangle_{\mathcal{M}} &= \int_{\mathcal{M}} \int_{\mathcal{M}} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t}\right) u(\mathbf{x})(u(\mathbf{x}) - u(\mathbf{y})) \mathrm{d}\mu_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{x}} \\
&= -\int_{\mathcal{M}} \int_{\mathcal{M}} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t}\right) u(\mathbf{y})(u(\mathbf{x}) - u(\mathbf{y})) \mathrm{d}\mu_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{x}} \\
&= \frac{1}{2} \int_{\mathcal{M}} \int_{\mathcal{M}} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t}\right) (u(\mathbf{x}) - u(\mathbf{y}))^2 \mathrm{d}\mu_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{x}}.
\end{aligned}
$$

Now, we turn to prove Theorem 7.2.

Proof. of Theorem 7.2

By Theorem 7.1 and the Poincaré inequality, there exists a constant $C > 0$, such that

$$\int_{\mathcal{M}} (v(\mathbf{x}) - \bar{v})^2 \mathrm{d}\mu_{\mathbf{x}} \leq \frac{CC_t}{t} \int_{\mathcal{M}} \int_{\mathcal{M}} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t}\right) (u(\mathbf{x}) - u(\mathbf{y}))^2 \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\mu_{\mathbf{y}}$$

where $\bar{v} = \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} v(\mathbf{x}) \mathrm{d}\mu_{\mathbf{x}}$ and

$$v(\mathbf{x}) = \frac{C_t}{w_t(\mathbf{x})} \int_{\mathcal{M}} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t}\right) u(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}},$$

At the same time, we have

$$\begin{aligned}
|\mathcal{M}||\bar{v}| &= \left| \int_{\mathcal{M}} v(\mathbf{x}) \mathrm{d}\mu_{\mathbf{x}} \right| \\
&= \left| \int_{\mathcal{M}} \int_{\mathcal{M}} \frac{C_t}{w_t(\mathbf{x})} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t}\right) (u(\mathbf{y}) - u(\mathbf{x})) \mathrm{d}\mu_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{x}} \right| \\
&\leq \left(\int_{\mathcal{M}} \int_{\mathcal{M}} \frac{C_t}{w_t(\mathbf{x})} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t}\right) \mathrm{d}\mu_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{x}} \right)^{1/2} \\
&\quad \left(\int_{\mathcal{M}} \int_{\mathcal{M}} \frac{C_t}{w_t(\mathbf{x})} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t}\right) (u(\mathbf{y}) - u(\mathbf{x}))^2 \mathrm{d}\mu_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{x}} \right)^{1/2} \\
&\leq C|\mathcal{M}|^{1/2} \left(C_t \int_{\mathcal{M}} \int_{\mathcal{M}} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t}\right) (u(\mathbf{y}) - u(\mathbf{x}))^2 \mathrm{d}\mu_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{x}} \right)^{1/2}
\end{aligned}$$

where the second equality comes from $\int_{\mathcal{M}} u(\mathbf{x}) \mathrm{d}\mu_{\mathbf{x}} = 0$. This enables us to upper bound the L_2 norm of v as follows. For t sufficiently small,

$$\begin{aligned}
\int_{\mathcal{M}} (v(\mathbf{x}))^2 \mathrm{d}\mu_{\mathbf{x}} &\leq 2 \int_M (v(\mathbf{x}) - \bar{v})^2 \mathrm{d}\mathbf{x} + 2 \int_M \bar{v}^2 \mathrm{d}\mu_{\mathbf{x}} \\
&\leq \frac{CC_t}{t} \int_{\mathcal{M}} \int_{\mathcal{M}} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t}\right) (u(\mathbf{x}) - u(\mathbf{y}))^2 \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\mu_{\mathbf{y}}
\end{aligned}$$

Let $\delta = \frac{w_{\min}}{2w_{\max} + w_{\min}}$ where $w_{\min} = \min_{\mathbf{x}} w_t(\mathbf{x})$ and $w_{\max} = \max_{\mathbf{x}} w_t(\mathbf{x})$. If u is smooth and close to its smoothed version v, in particular,

$$\int_{\mathcal{M}} |v(\mathbf{x})|^2 \mathrm{d}\mu_{\mathbf{x}} \geq \delta^2 \int_{\mathcal{M}} |u(\mathbf{x})|^2 \mathrm{d}\mu_{\mathbf{x}}, \tag{7.4}$$

then the theorem is proved.

Now consider the case where (7.4) does not hold. Note that we now have

$$\begin{aligned}
\|u - v\|_{L^2(\mathcal{M})} &\geq \|u\|_{L_2(\mathcal{M})} - \|v\|_{L_2(\mathcal{M})} > (1 - \delta)\|u\|_{L_2(\mathcal{M})} \\
&> \frac{1 - \delta}{\delta} \|v\|_{L_2(\mathcal{M})} = \frac{2w_{\max}}{w_{\min}} \|v\|_{L_2(\mathcal{M})}.
\end{aligned}$$

Then we have

$$\frac{C_t}{t} \int_{\mathcal{M}} \int_{\mathcal{M}} R\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right) (u(\mathbf{x})-u(\mathbf{y}))^2 \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\mu_{\mathbf{y}}$$

$$= \frac{2C_t}{t} \int_{\mathcal{M}} u(\mathbf{x}) \int_{\mathcal{M}} R\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right) (u(\mathbf{x})-u(\mathbf{y})) \mathrm{d}\mu_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{x}}$$

$$= \frac{2}{t}\left(\int_{\mathcal{M}} u^2(\mathbf{x})w(\mathbf{x})\mathrm{d}\mu_{\mathbf{x}} - \int_{\mathcal{M}} u(\mathbf{x})v(\mathbf{x})w(\mathbf{x})\mathrm{d}\mu_{\mathbf{x}}\right)$$

$$= \frac{2}{t}\left(\int_{\mathcal{M}} (u(\mathbf{x})-v(\mathbf{x}))^2 w(\mathbf{x})\mathrm{d}\mu_{\mathbf{x}} + \int_{\mathcal{M}} (u(\mathbf{x})-v(\mathbf{x}))v(\mathbf{x})w(\mathbf{x})\mathrm{d}\mu_{\mathbf{x}}\right)$$

$$\geq \frac{2}{t}\int_{\mathcal{M}} (u(\mathbf{x})-v(\mathbf{x}))^2 w(\mathbf{x})\mathrm{d}\mu_{\mathbf{x}} - \frac{2}{t}\left(\int_{\mathcal{M}} v^2(\mathbf{x})w(\mathbf{x})\mathrm{d}\mu_{\mathbf{x}}\right)^{1/2}\left(\int_{\mathcal{M}} (u(\mathbf{x})-v(\mathbf{x}))^2 w(\mathbf{x})\mathrm{d}\mu_{\mathbf{x}}\right)^{1/2}$$

$$\geq \frac{2w_{\min}}{t}\int_{\mathcal{M}} (u(\mathbf{x})-v(\mathbf{x}))^2 \mathrm{d}\mu_{\mathbf{x}} - \frac{2w_{\max}}{t}\left(\int_{\mathcal{M}} v^2(\mathbf{x})\mathrm{d}\mu_{\mathbf{x}}\right)^{1/2}\left(\int_{\mathcal{M}} (u(\mathbf{x})-v(\mathbf{x}))^2 \mathrm{d}\mu_{\mathbf{x}}\right)^{1/2}$$

$$\geq \frac{w_{\min}}{t}\int_{\mathcal{M}} (u(\mathbf{x})-v(\mathbf{x}))^2 \mathrm{d}\mu_{\mathbf{x}} \geq \frac{w_{\min}}{t}(1-\delta)^2 \int_{\mathcal{M}} u^2(\mathbf{x})\mathrm{d}\mu_{\mathbf{x}}.$$

This completes the proof for the theorem. □

7.1   Proof of Theorem 4.1

With Theorem 7.1 and 7.2, the proof of Theorem 4.1 is straightforward.

Proof. of Theorem 4.1
   Using Theorem 7.2, we have

$$\|u\|^2_{L^2(\mathcal{M})} \leq C \langle u, L_t u\rangle = C \int_{\mathcal{M}} u(\mathbf{x})(r(\mathbf{x}) - \bar{r})\mathrm{d}\mu_{\mathbf{x}} \tag{7.5}$$
$$\leq C\|u\|_{L^2(\mathcal{M})}\|r\|_{L^2(\mathcal{M})}.$$

To show the last inequality, we use the fact that

$$|\bar{r}| = \frac{1}{|\mathcal{M}|}\left|\int_{\mathcal{M}} r(\mathbf{x})\mathrm{d}\mu_{\mathbf{x}}\right| \leq C\|r\|_{L^2(\mathcal{M})}.$$

This inequality (7.5) implies that

$$\|u\|_{L^2(\mathcal{M})} \leq C\|r\|_{L^2(\mathcal{M})}.$$

Now we turn to estimate $\|\nabla u\|_{L^2(\mathcal{M})}$. Notice that we have the following expression for u, since u satisfies the integral equation (4.2).

$$u(\mathbf{x}) = v(\mathbf{x}) + \frac{t}{w_t(\mathbf{x})}(r(\mathbf{x}) - \bar{r}),$$

where

$$v(\mathbf{x}) = \frac{1}{w_t(\mathbf{x})}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}, \quad w_t(\mathbf{x}) = \int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}.$$

By Theorem 7.1, we have

$$
\begin{aligned}
\|\nabla u\|_{L^2(\mathcal{M})}^2 \leq & 2\|\nabla v\|_{L^2(\mathcal{M})}^2 + 2t^2 \left\| \nabla \left(\frac{r(\mathbf{x}) - \bar{r}}{w_t(\mathbf{x})} \right) \right\|_{L^2(\mathcal{M})}^2 \\
\leq & C \left\langle u, L_t u \right\rangle + Ct\|r\|_{L^2(\mathcal{M})}^2 + Ct^2\|\nabla r\|_{L^2(\mathcal{M})}^2 \\
\leq & C\|u\|_{L^2(\mathcal{M})}\|r\|_{L^2(\mathcal{M})} + Ct\|r\|_{L^2(\mathcal{M})}^2 + Ct^2\|\nabla r\|_{L^2(\mathcal{M})}^2 \\
\leq & C\|r\|_{L^2(\mathcal{M})}^2 + Ct^2\|\nabla r\|_{L^2(\mathcal{M})}^2 \\
\leq & C \left(\|r\|_{L^2(\mathcal{M})} + t\|\nabla r\|_{L^2(\mathcal{M})} \right)^2 .
\end{aligned}
$$

This completes the proof. $\qquad\square$

7.2 Proof of Theorem 4.4

The proof of Theorem 4.4 is more involved.

Proof. First, we denote

$$
r(\mathbf{x}) = \int_{\partial \mathcal{M}} \mathbf{b}(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \bar{R}_t(\mathbf{x}, \mathbf{y}) \mathrm{d}\tau_{\mathbf{y}},
$$
$$
\bar{r} = \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} \left(\int_{\partial \mathcal{M}} \mathbf{b}(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \bar{R}_t(\mathbf{x}, \mathbf{y}) \mathrm{d}\tau_{\mathbf{y}} \right) \mathrm{d}\mathbf{x}.
$$

where $|\mathcal{M}| = \int_{\mathcal{M}} \mathrm{d}\mu_{\mathbf{y}}$.

The key point of the proof is to show that

$$
\left| \int_{\mathcal{M}} u(\mathbf{x}) \left(r(\mathbf{x}) - \bar{r} \right) \mathrm{d}\mu_{\mathbf{x}} \right| \leq C\sqrt{t}\, \|\mathbf{b}\|_{H^1(\mathcal{M})} \|u\|_{H^1(\mathcal{M})}. \tag{7.6}
$$

First, notice that

$$
|\bar{r}| \leq C\sqrt{t}\, \|\mathbf{b}\|_{L^2(\partial \mathcal{M})} \leq C\sqrt{t}\, \|\mathbf{b}\|_{H^1(\mathcal{M})}.
$$

Then it is sufficient to show that

$$
\left| \int_{\mathcal{M}} u(\mathbf{x}) \left(\int_{\partial \mathcal{M}} \mathbf{b}(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \bar{R}_t(\mathbf{x}, \mathbf{y}) \mathrm{d}\tau_{\mathbf{y}} \right) \mathrm{d}\mu_{\mathbf{x}} \right| \leq C\sqrt{t}\, \|\mathbf{b}\|_{H^1(\mathcal{M})} \|u\|_{H^1(\mathcal{M})}. \tag{7.7}
$$

Direct calculation gives that

$$
|2t\nabla \bar{\bar{R}}_t(\mathbf{x}, \mathbf{y}) - (\mathbf{x} - \mathbf{y})\bar{R}_t(\mathbf{x}, \mathbf{y})| \leq C|\mathbf{x} - \mathbf{y}|^2 \bar{R}_t(\mathbf{x}, \mathbf{y}),
$$

where $\bar{\bar{R}}_t(\mathbf{x}, \mathbf{y}) = C_t \bar{\bar{R}} \left(\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4t} \right)$ and $\bar{\bar{R}}(r) = \int_r^\infty \bar{R}(s)\mathrm{d}s$. This implies that

$$
\left| \int_{\mathcal{M}} u(\mathbf{x}) \int_{\partial \mathcal{M}} \mathbf{b}(\mathbf{y}) \left((\mathbf{x} - \mathbf{y})\bar{R}_t(\mathbf{x}, \mathbf{y}) + 2t\nabla \bar{\bar{R}}_t(\mathbf{x}, \mathbf{y}) \right) \mathrm{d}\tau_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{x}} \right| \tag{7.8}
$$
$$
\leq C \int_{\mathcal{M}} |u(\mathbf{x})| \int_{\partial \mathcal{M}} |\mathbf{b}(\mathbf{y})||\mathbf{x} - \mathbf{y}|^2 \bar{R}_t(\mathbf{x}, \mathbf{y}) \mathrm{d}\tau_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{x}}
$$
$$
\leq Ct\|\mathbf{b}\|_{L^2(\partial \mathcal{M})} \left(\int_{\partial \mathcal{M}} \left(\int_{\mathcal{M}} \bar{R}_t(\mathbf{x}, \mathbf{y})\mathrm{d}\mu_{\mathbf{x}} \right) \left(\int_{\mathcal{M}} |u(\mathbf{x})|^2 \bar{R}_t(\mathbf{x}, \mathbf{y})\mathrm{d}\mu_{\mathbf{x}} \right) \mathrm{d}\tau_{\mathbf{y}} \right)^{1/2}
$$
$$
\leq Ct\|\mathbf{b}\|_{H^1(\mathcal{M})} \left(\int_{\mathcal{M}} |u(\mathbf{x})|^2 \left(\int_{\partial \mathcal{M}} \bar{R}_t(\mathbf{x}, \mathbf{y})\mathrm{d}\tau_{\mathbf{y}} \right) \mathrm{d}\mu_{\mathbf{x}} \right)^{1/2}
$$
$$
\leq Ct^{3/4}\|\mathbf{b}\|_{H^1(\mathcal{M})} \|u\|_{L^2(\mathcal{M})}.
$$

22

On the other hand, using the Gauss integral formula, we have

$$\int_{\mathcal{M}} u(\mathbf{x}) \int_{\partial\mathcal{M}} \mathbf{b}(\mathbf{y}) \cdot \nabla \bar{\bar{R}}_t(\mathbf{x},\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{x}} \tag{7.9}$$

$$= \int_{\partial\mathcal{M}} \int_{\mathcal{M}} u(\mathbf{x}) T_{\mathbf{x}}(\mathbf{b}(\mathbf{y})) \cdot \nabla \bar{\bar{R}}_t(\mathbf{x},\mathbf{y}) \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\tau_{\mathbf{y}}$$

$$= \int_{\partial\mathcal{M}} \int_{\partial\mathcal{M}} \mathbf{n}(\mathbf{x}) \cdot T_{\mathbf{x}}(\mathbf{b}(\mathbf{y})) u(\mathbf{x}) \bar{\bar{R}}_t(\mathbf{x},\mathbf{y}) \mathrm{d}\tau_{\mathbf{x}} \mathrm{d}\tau_{\mathbf{y}}$$

$$- \int_{\partial\mathcal{M}} \int_{\mathcal{M}} \mathrm{div}_{\mathbf{x}}[u(\mathbf{x}) T_{\mathbf{x}}(\mathbf{b}(\mathbf{y}))] \bar{\bar{R}}_t(\mathbf{x},\mathbf{y}) \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\tau_{\mathbf{y}}.$$

Here $T_{\mathbf{x}}$ is the projection operator to the tangent space on $\mathbf{x}$. To get the first equality, we use the fact that $\nabla \bar{\bar{R}}_t(\mathbf{x},\mathbf{y})$ belongs to the tangent space on $\mathbf{x}$, such that $\mathbf{b}(\mathbf{y}) \cdot \nabla \bar{\bar{R}}_t(\mathbf{x},\mathbf{y}) = T_{\mathbf{x}}(\mathbf{b}(\mathbf{y})) \cdot \nabla \bar{\bar{R}}_t(\mathbf{x},\mathbf{y})$ and $\mathbf{n}(\mathbf{x}) \cdot T_{\mathbf{x}}(\mathbf{b}(\mathbf{y})) = \mathbf{n}(\mathbf{x}) \cdot \mathbf{b}(\mathbf{y})$ where $\mathbf{n}(\mathbf{x})$ is the out normal of $\partial\mathcal{M}$ at $\mathbf{x} \in \partial\mathcal{M}$.

For the first term, we have

$$\left| \int_{\partial\mathcal{M}} \int_{\partial\mathcal{M}} \mathbf{n}(\mathbf{x}) \cdot T_{\mathbf{x}}(\mathbf{b}(\mathbf{y})) u(\mathbf{x}) \bar{\bar{R}}_t(\mathbf{x},\mathbf{y}) \mathrm{d}\tau_{\mathbf{x}} \mathrm{d}\tau_{\mathbf{y}} \right| \tag{7.10}$$

$$= \left| \int_{\partial\mathcal{M}} \int_{\partial\mathcal{M}} \mathbf{n}(\mathbf{x}) \cdot \mathbf{b}(\mathbf{y}) u(\mathbf{x}) \bar{\bar{R}}_t(\mathbf{x},\mathbf{y}) \mathrm{d}\tau_{\mathbf{x}} \mathrm{d}\tau_{\mathbf{y}} \right|$$

$$\leq C \|\mathbf{b}\|_{L^2(\partial\mathcal{M})} \left(\int_{\partial\mathcal{M}} \left(\int_{\partial\mathcal{M}} |u(\mathbf{x})| \bar{\bar{R}}_t(\mathbf{x},\mathbf{y}) \mathrm{d}\tau_{\mathbf{x}} \right)^2 \mathrm{d}\tau_{\mathbf{y}} \right)^{1/2}$$

$$\leq C \|\mathbf{b}\|_{H^1(\mathcal{M})} \left(\int_{\partial\mathcal{M}} \left(\int_{\partial\mathcal{M}} \bar{\bar{R}}_t(\mathbf{x},\mathbf{y}) \mathrm{d}\tau_{\mathbf{x}} \right) \left(\int_{\partial\mathcal{M}} |u(\mathbf{x})|^2 \bar{\bar{R}}_t(\mathbf{x},\mathbf{y}) \mathrm{d}\tau_{\mathbf{x}} \right) \mathrm{d}\tau_{\mathbf{y}} \right)^{1/2}$$

$$\leq C t^{-1/2} \|\mathbf{b}\|_{H^1(\mathcal{M})} \|u\|_{L^2(\partial\mathcal{M})} \leq C t^{-1/2} \|\mathbf{b}\|_{H^1(\mathcal{M})} \|u\|_{H^1(\mathcal{M})}.$$

We can also bound the second term on the right hand side of (7.9). By using the assumption that $\mathcal{M} \in C^{\infty}$, we have

$$|\mathrm{div}_{\mathbf{x}}[u(\mathbf{x}) T_{\mathbf{x}}(\mathbf{b}(\mathbf{y}))]|$$

$$\leq |\nabla u(\mathbf{x})||T_{\mathbf{x}}(\mathbf{b}(\mathbf{y}))||| + |u(\mathbf{x})||\mathrm{div}_{\mathbf{x}}[T_{\mathbf{x}}(\mathbf{b}(\mathbf{y}))]||| + |\nabla||u(\mathbf{x}) T_{\mathbf{x}}(\mathbf{b}(\mathbf{y}))|$$

$$\leq C(|\nabla u(\mathbf{x})| + |u(\mathbf{x})|)|\mathbf{b}(\mathbf{y})|$$

where the constant C depends on the curvature of the manifold $\mathcal{M}$.

Then, we have

$$\left| \int_{\partial\mathcal{M}} \int_{\mathcal{M}} \mathrm{div}_{\mathbf{x}}[u(\mathbf{x}) T_{\mathbf{x}}(\mathbf{b}(\mathbf{y}))] \bar{\bar{R}}_t(\mathbf{x},\mathbf{y}) \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\tau_{\mathbf{y}} \right| \tag{7.11}$$

$$\leq C \int_{\partial\mathcal{M}} \mathbf{b}(\mathbf{y}) \int_{\mathcal{M}} (|\nabla u(\mathbf{x})| + |u(\mathbf{x})|) \bar{\bar{R}}_t(\mathbf{x},\mathbf{y}) \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\tau_{\mathbf{y}}$$

$$\leq C \|\mathbf{b}\|_{L^2(\partial\mathcal{M})} \left(\int_{\mathcal{M}} (|\nabla u(\mathbf{x})|^2 + |u(\mathbf{x})|^2) \left(\int_{\partial\mathcal{M}} \bar{\bar{R}}_t(\mathbf{x},\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}} \right) \mathrm{d}\mu_{\mathbf{x}} \right)^{1/2}$$

$$\leq C t^{-1/4} \|\mathbf{b}\|_{H^1(\mathcal{M})} \|u\|_{H^1(\mathcal{M})}.$$

Then, the inequality (7.7) is obtained from (7.8), (7.9), (7.10) and (7.11). Now, using Theorem 7.2, we have

$$\|u\|_{L^2(\mathcal{M})}^2 \leq C \int_{\mathcal{M}} u(\mathbf{x}) L_t u(\mathbf{x}) \mathrm{d}\mu_{\mathbf{x}} \leq C\sqrt{t} \|\mathbf{b}\|_{H^1(\mathcal{M})} \|u\|_{H^1(\mathcal{M})}. \tag{7.12}$$

Note $r(\mathbf{x}) = \int_{\partial\mathcal{M}} (\mathbf{x} - \mathbf{y}) \cdot \mathbf{b}(\mathbf{y}) \bar{R}_t(\mathbf{x}, \mathbf{y}) \mathrm{d}\tau_{\mathbf{y}}$. Direct calculation gives us that

$$\|r(\mathbf{x})\|_{L^2(\mathcal{M})} \le Ct^{1/4}\|\mathbf{b}\|_{H^1(\mathcal{M})}, \text{ and}$$
$$\|\nabla r(\mathbf{x})\|_{L^2(\mathcal{M})} \le Ct^{-1/4}\|\mathbf{b}\|_{H^1(\mathcal{M})}.$$

The integral equation $L_t u = r - \bar{r}$ gives that

$$u(\mathbf{x}) = v(\mathbf{x}) + \frac{t}{w_t(\mathbf{x})}\left(r(\mathbf{x}) - \bar{r}\right)$$

where

$$v(\mathbf{x}) = \frac{1}{w_t(\mathbf{x})} \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) u(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}}, \quad w_t(\mathbf{x}) = \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) \mathrm{d}\mu_{\mathbf{y}}.$$

By Theorem 7.1, we have

$$\|\nabla u\|_{L^2(\mathcal{M})}^2 \tag{7.13}$$
$$\le 2\|\nabla v\|_{L^2(\mathcal{M})}^2 + 2t^2 \left\| \nabla\left(\frac{r(\mathbf{x}) - \bar{r}}{w_t(\mathbf{x})}\right) \right\|_{L^2(\mathcal{M})}^2$$
$$\le C \int_{\mathcal{M}} u(\mathbf{x}) L_t u(\mathbf{x}) \mathrm{d}\mu_{\mathbf{x}} + Ct\|r\|_{L^2(\mathcal{M})}^2 + Ct^2\|\nabla r\|_{L^2(\mathcal{M})}^2$$
$$\le C\sqrt{t}\,\|\mathbf{b}\|_{H^1(\mathcal{M})}\|u\|_{H^1(\mathcal{M})} + Ct\|r\|_{L^2(\mathcal{M})}^2 + Ct^2\|\nabla r\|_{L^2(\mathcal{M})}^2$$
$$\le C\|\mathbf{b}\|_{H^1(\mathcal{M})}\left(\sqrt{t}\|u\|_{H^1(\mathcal{M})} + Ct^{3/2}\right).$$

Using (7.12) and (7.13), we have

$$\|u\|_{H^1(\mathcal{M})}^2 \le C\|\mathbf{b}\|_{H^1(\mathcal{M})}\left(\sqrt{t}\|u\|_{H^1(\mathcal{M})} + Ct^{3/2}\right),$$

which proves the theorem. $\qquad\square$

8 Discussion and Future Work

We have proved the convergence of the point integral method for Poisson equations on the submanifolds isometrically embedded in Euclidean spaces. Our analysis shows that the convergence rate of PIM is $h^{1/4}(P, \mathbf{V}, \mathcal{M})$ in H^1 norm. However, our experimental results in [28], show the empirical convergence rate is about linear. Indeed, there are places in our analysis where we believe the error bounds can be improved.

On the other hand, the quadrature rule we used in the point integral method is of low accuracy. If we have more information, such as the local mesh or local hypersurface, we could use high order quadrature rule to improve the accuracy of the point integral method.

Based on the convergence result in this paper, we are able to show that the spectra of the graph Laplacian with a proper normalization converge to the spectra of $\Delta_{\mathcal{M}}$ with the Neumann boundary condition. Moreover, we can obtain an estimate of the rate of the spectral convergence. The point integral method also applies to Poisson equation with Dirichlet boundary. And we can also show the convergence of the point integral method for the Dirichlet boundary. These results will be reported in the subsequent papers.
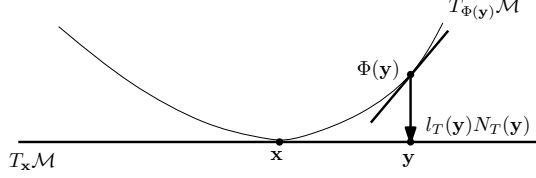
Figure 1: Parametrization for a neighborhood of a point on $\mathcal{M}$.

A Proof of Proposition 5.1

To prove the proposition, we first cite a few results from Riemannian geometry on isometric embeddings. For a submanifold $\mathcal{M}$ embedded in $\mathbb{R}^d$, let $d_{\mathcal{M}} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ be the geodesic distance on $\mathcal{M}$, and $T_{\mathbf{x}}\mathcal{M}$ and $N_{\mathbf{x}}\mathcal{M}$ be the tangent space and the normal space of $\mathcal{M}$ at point $\mathbf{x} \in \mathcal{M}$ respectively.

Lemma A.1. *(eg [13]) Assume $\mathcal{M}$ is a submanifold isometrically embedded in $\mathbb{R}^d$ with reach $\sigma > 0$. For any two $\mathbf{x}, \mathbf{y}$ on $\mathcal{M}$ with $|\mathbf{x} - \mathbf{y}| \leq \sigma/2$,*

$$|\mathbf{x} - \mathbf{y}| \leq d_{\mathcal{M}}(\mathbf{x}, \mathbf{y}) \leq |\mathbf{x} - \mathbf{y}|(1 + \frac{2|\mathbf{x} - \mathbf{y}|^2}{\sigma^2}).$$

Lemma A.2. *(eg [8]) Assume $\mathcal{M}$ is a submanifold isometrically embedded in $\mathbb{R}^d$ with reach $\sigma > 0$. For any two $\mathbf{x}, \mathbf{y}$ on $\mathcal{M}$ with $|\mathbf{x} - \mathbf{y}| \leq \sigma/2$,*

$$\cos \angle T_{\mathbf{x}}\mathcal{M}, T_{\mathbf{y}}\mathcal{M} \leq 1 - \frac{2|\mathbf{x} - \mathbf{y}|^2}{\sigma^2}.$$

Lemma A.3. *(eg [33]) Assume $\mathcal{M}$ is a submanifold isometrically embedded in $\mathbb{R}^d$ with reach $\sigma > 0$. Let N be any local normal vector field around a point $\mathbf{x} \in \mathcal{M}$. Then for any tangent vector $Y \in T_{\mathbf{x}}\mathcal{M}$*

$$\frac{< Y, D_Y N >}{< Y, Y >} \leq \frac{1}{\sigma}$$

where D and $< \cdot, \cdot >$ are the standard connection and the standard inner product in $\mathbb{R}^d$.

In what follows, assume the hypotheses on $\mathcal{M}$ and $\partial\mathcal{M}$ in Proposition 5.1 hold. We prove the following two lemmas which bound the distortion of certain parametrization, which are used to build the parametrization stated in Proposition 5.1.

For a point $\mathbf{x} \in \mathcal{M}$,, let $U_\rho = B_{\mathbf{x}}(\rho\sigma) \cap \mathcal{M}$ with $\rho \leq 0.2$. We define the following projection map $\Psi : U_\rho \to T_{\mathbf{x}}\mathcal{M} = \mathbb{R}^k$ as the restriction to U_ρ of the projection of $\mathbb{R}^d$ onto $T_{\mathbf{x}}\mathcal{M}$. It is easy to verify that Ψ is one-to-one. Then $\Phi = \Psi^{-1} : \Psi(U_\rho) \to U_\rho$ is a parametrization of U_ρ. See Figure 1. We have the following lemma which bounds the distortion of this parametrization.

Lemma A.4. *For any point $\mathbf{y} \in \Psi(U_\rho)$ and any $Y \in T_{\mathbf{y}}(T_{\mathbf{x}}\mathcal{M})$ for any $\rho \leq 0.2$,*

$$|Y| \leq |D_Y \Phi(\mathbf{y})| \leq \frac{1}{1 - 2\rho^2}|Y|.$$

Proof. We have $\Phi(\mathbf{y}) = \mathbf{y} - l_T(\mathbf{y})N_T(\mathbf{y})$ where $N_T(\mathbf{y}) \perp T_{\mathbf{x}}\mathcal{M}$ for any $\mathbf{y}$ and $l_T(\mathbf{y}) = |\mathbf{y} - \Phi(\mathbf{y})|$. So $D_Y N_T(\mathbf{y}) \perp T_{\mathbf{x}}\mathcal{M}$ for any $\mathbf{y}$ and any $Y \in T_{\mathbf{y}}(T_{\mathbf{x}}\mathcal{M})$. Since $D_Y \Phi = Y - N_T(D_Y l_T) - l_T(D_Y N_T)$, the projection of $D_Y \Phi$ to $T_{\mathbf{x}}\mathcal{M}$ is Y. At the same time, $D_Y \Phi$ is on $T_{\Phi(\mathbf{y})}\mathcal{M}$. Since $|\mathbf{x} - \Phi(\mathbf{y})| \leq \rho\sigma$, from Lemma A.2, $\cos \angle T_{\mathbf{x}}\mathcal{M}, T_{\Phi(\mathbf{y})}\mathcal{M} \leq 1 - 2\rho^2$. This proves the lemma. $\qquad \square$
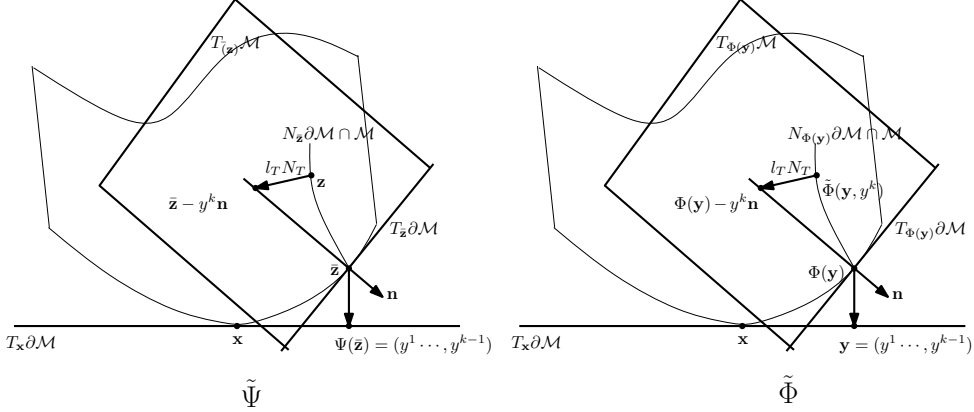
Figure 2: Parametrization for a neighborhood of a point on $\partial\mathcal{M}$

To ensure the convexity of the parameter domain Ω in Proposition 5.1, We need a different parametrization for the points near the boundary. For a point $\mathbf{x} \in \partial\mathcal{M}$, let $U_\rho = B_\mathbf{x}(\rho\sigma) \cap \mathcal{M}$ with $\rho \leq 0.1$. We construct a map $\tilde{\Psi} : U_\rho \to T_\mathbf{x}\partial\mathcal{M} \times \mathbb{R} = \mathbb{R}^k$ as follows. For any point $\mathbf{z} \in U_\rho$, let $\bar{\mathbf{z}}$ be the closest point on $\partial\mathcal{M}$ to $\mathbf{z}$. Such $\bar{\mathbf{z}}$ is unique. Let $\mathbf{n}$ be the outward normal of $\partial\mathcal{M}$ at $\bar{\mathbf{z}}$. The projection P of $\mathbb{R}^d$ onto $T_{\bar{\mathbf{z}}}\mathcal{M}$ maps $\mathbf{z}$ to a point on the line ℓ passing through $\bar{\mathbf{z}}$ with the direction $\mathbf{n}$. In fact, P projects $N_{\bar{\mathbf{z}}}\partial\mathcal{M}$ onto the line ℓ. If let $V_{\rho_1} = N_{\bar{\mathbf{z}}}\partial\mathcal{M} \cap B_{\bar{\mathbf{z}}}(\rho_1\sigma) \cap \mathcal{M}$ with $\rho_1 \leq 0.2$, P maps V_{ρ_1} to the line ℓ in the one-to-one manner. Let $y^k = -(P(\mathbf{z}) - \bar{\mathbf{z}}) \cdot \mathbf{n}$. Think of $\partial\mathcal{M}$ as a submanifold. It is isometrically embedded in $\mathbb{R}^d$ as is $\mathcal{M}$. As $|\bar{\mathbf{z}} - \mathbf{x}| \leq 2|\mathbf{x} - \mathbf{z}| \leq 2\rho\sigma$, we apply Lemma A.4 by replacing $\mathcal{M}$ with $\partial\mathcal{M}$ and obtain the map Ψ that maps $\bar{\mathbf{z}}$ onto $T_\mathbf{x}\partial\mathcal{M}$. Define $\tilde{\Psi}(\mathbf{z}) = (\Psi(\bar{\mathbf{z}}), y^k)$. Since both $P|_{V_{\rho_1}}$ and Ψ are one-to-one, so is $\tilde{\Psi}$. Then $\tilde{\Phi} = \tilde{\Psi}^{-1} : \tilde{\Psi}(U_\rho) \to U_\rho$ is a parametrization of U_ρ. See Figure 2. We have the following lemma which bounds the distortion of this parametrization $\tilde{\Phi}$.

Lemma A.5. *For any point $(\mathbf{y}, y^k) \in \tilde{\Psi}(U_\rho)$ with $\rho \leq 0.1$ and any tangent vector Y at $(\mathbf{y}, y^k)$,*

$$(1 - 2\rho)|Y| \leq |D_Y\tilde{\Phi}(\mathbf{y}, y^k)| \leq (1 + 2\rho)|Y|.$$

Proof. Let $\bar{\mathbf{y}} = \Phi(\mathbf{y}) - y^k\mathbf{n}$. We have $\tilde{\Phi}(\mathbf{y}, y^k) = \Phi(\mathbf{y}) - y^k\mathbf{n}(\Phi(\mathbf{y})) - l_T(\bar{\mathbf{y}})N_T(\bar{\mathbf{y}})$ where $N_T(\bar{\mathbf{y}}) \perp T_{\Phi(\mathbf{y})}\mathcal{M}$. See Figure 2. We have

$$D_Y\tilde{\Phi}(\mathbf{y}, y^k) = D_Y\Phi - y^k D_Y\mathbf{n} - \mathbf{n}D_Y y^k - N_T D_Y l_T - l_T D_Y N_t.$$

Using the similar strategy of proving Lemma A.4, we consider the projection of $D_Y\tilde{\Phi}(\mathbf{y}, y^k)$ to the space $T_{\Phi}(\mathbf{y})M$ to which it is almost parallel. Denote P this projection map . We bound $P(D_Y\tilde{\Phi}(\mathbf{y}, y^k))$. Let $Y = (Y^1, \cdots, Y^k)$, $Y_1 = (Y^1, \cdots, Y^{k-1}, 0)$ and $Y_2 = (0, \cdots, 0, Y^k)$. We have $D_Y\tilde{\Phi}(\mathbf{y}, y^k) = D_{Y_1}\tilde{\Phi}(\mathbf{y}, y^k) + D_{Y_2}\tilde{\Phi}(\mathbf{y}, y^k)$. First consider each term involved in $D_{Y_1}\tilde{\Phi}(\mathbf{y}, y^k)$.

(i) $D_{Y_1}\Phi(\mathbf{y})$ is a vector in $T_{\Phi(\mathbf{y})}\partial\mathcal{M}$, thus $P(D_{Y_1}\Phi(\mathbf{y})) = D_{Y_1}\Phi(\mathbf{y})$. In addition, from Lemma A.4, $|Y_1| \leq |D_{Y_1}\Phi(\mathbf{y})| \leq \frac{1}{1-2\rho^2}|Y_1|$.

(ii) $D_{Y_1}\mathbf{n}(\mathbf{y}, y^k) = D_{D_{Y_1}\Phi}\mathbf{n}(\Phi(\mathbf{y}))$. First note that $\mathbf{n} \cdot D_{D_{Y_1}\Phi}\mathbf{n} = 0$. Second, from Lemma A.3, we have that the projection of $D_{D_{Y_1}\Phi}\mathbf{n}$ to the space $T_{\Phi(\mathbf{y})}\partial\mathcal{M}$ is upper bounded by $\frac{1}{\sigma}|D_{Y_1}\Phi|$. Since $|y^k| < \rho\sigma$, $|P(y^k D_{Y_1}\mathbf{n})| \leq \frac{\rho}{1-2\rho^2}|Y_1|$.

26

(iii) Consider $D_{Y_1} N_T(\mathbf{y}, y^k)$. We have $N_T \perp T_{\Phi(\mathbf{y})}\mathcal{M}$. Let $e_1, \cdots, e_k$ be the orthonormal basis of $T_{\Phi(\mathbf{y})}\mathcal{M}$ so that $D_{e_i} N_T \cdot e_j = 0$ for $i \neq j$. Locally extend $e_1, \cdots, e_k$ to be an orthonormal basis of $T\mathcal{M}$ in a neighborhood of $\Phi(\mathbf{y})$. We have for any e_i

$$
\begin{aligned}
\left| D_{Y_1} N_T(\mathbf{y}, y^k) \cdot e_i \right| &= \left| D_{D_{Y_1}\Phi} N_T(\bar{\mathbf{y}}) \cdot e_i \right| \\
&= \left| D_{(D_{Y_1}\Phi \cdot e_i)e_i} N_T(\bar{\mathbf{y}}) \cdot e_i \right| \\
&= \left| D_{(D_{Y_1}\Phi \cdot e_i)e_i} N_T(\bar{\mathbf{y}}) \cdot e_i \right| \\
&\leq \frac{1}{\sigma} |D_{Y_1}\Phi \cdot e_i|
\end{aligned}
$$

where the last inequality is due to Lemma A.3. Moreover, one can verify that $l_T(\bar{\mathbf{y}}) \leq \frac{\rho^2 \sigma}{2}$, which leads to

$$
|P(l_T D_{Y_1} N_t)| \leq \frac{\rho^2}{2} |D_{Y_1}\Phi| \leq \frac{\rho^2}{2(1 - 2\rho^2)} |Y_1|
$$

(iv) It is obvious that $\mathbf{n} D_{Y_1} y^k = P(N_T D_{Y_1} l_T) = 0$.

Next consider each term involved in $D_{Y_2}\tilde{\Phi}(\mathbf{y}, y^k)$.

(i) $\mathbf{n} D_{Y_2} y^k = Y^k \mathbf{n}$, which lies on $T_{\Phi(\mathbf{y})}\mathcal{M}$. Moreover $\mathbf{n} \perp D_{Y_1}\Phi(\mathbf{y})$.

(ii) As $N_T(\mathbf{y}, y^k)$ remains perpendicular to $T_{\Phi(\mathbf{y})}\mathcal{M}$ if we only vary y^k, we have

$$
P(D_{Y_2} N_t(\mathbf{y}, y^k)) = 0.
$$

(iii) For the remaining terms, we have $D_{Y_2}\Phi(\mathbf{y}) = y^k D_{Y_2}\mathbf{n} = P(N_T D_{Y_2} l_T) = 0$.

On the other hand, we hand $D_Y \tilde{\Phi}(\mathbf{y}, y^k)$ lie in the tangent space $T_{\tilde{\Phi}(\mathbf{y}, y^k)}\mathcal{M}$, and

$$
\cos \angle T_{\tilde{\Phi}(\mathbf{y}, y^k)}\mathcal{M}, T_{\Phi(\mathbf{y})}\mathcal{M} \leq 1 - 2\rho^2.
$$

Putting everything together, we have

$$
|Y| - \frac{\rho^2 + 2\rho}{2(1 - 2\rho^2)} |Y_1| \leq D_Y \tilde{\Phi}(\mathbf{y}, y^k) \leq \frac{1}{(1 - 2\rho^2)^2} |Y| + \frac{\rho^2 + 2\rho}{2(1 - 2\rho^2)^2} |Y_1|.
$$

This proves the lemma. $\qquad \square$

Now we are ready to prove Proposition 5.1

Proof. of Proposition 5.1
First consider the case where $d(\mathbf{x}, \partial\mathcal{M}) > \frac{\rho}{2}\sigma$. Set $U' = B_{\mathbf{x}}(\frac{\rho}{2}\sigma) \cap \mathcal{M}$, and parametrize U' using map $\Phi : \Psi(U') \to U'$. Since for any $\mathbf{y} \in \partial U'$, $|\mathbf{x} - \mathbf{y}| = \frac{\rho}{2}\sigma$, from Lemma A.4, we have that $B_{\Phi^{-1}(\mathbf{x})}(\frac{\rho}{2(1+\rho)}\sigma)$ is contained in $\Psi(U')$. Set $\Omega = B_{\Phi^{-1}(\mathbf{x})}(\frac{\rho}{2(1+\rho)})$ and $U = \Phi(\Omega)$. This shows the parametrization $\Phi : \Omega \to U$ satisfies the condition (i). By Lemma A.4 and Lemma A.1, it is easy to verify that Φ satisfies the other three conditions.

Next consider the case where $d(\mathbf{x}, \partial\mathcal{M}) \leq \frac{\rho}{2}\sigma$. Let $\bar{\mathbf{x}}$ be the closest point on $\partial\mathcal{M}$ to $\mathbf{x}$. Set $U' = B_{\bar{\mathbf{x}}}(\rho\sigma) \cap \mathcal{M}$ and parametrize U' using map $\tilde{\Phi} : \tilde{\Psi}(U') \to U'$. By Lemma A.5, $\tilde{\Psi}(U')$ contains half of the ball $B_{\tilde{\Phi}^{-1}(\bar{\mathbf{x}})}(\frac{\rho\sigma}{1+2\rho})$. Let Ω be that half ball and $U = \tilde{\Phi}(\Omega)$. It is easy to verify that the parametrization $\tilde{\Phi} : \Omega \to U$ satisfies the condition (iii) and (iv). To see (i), note that $|\mathbf{x} - \bar{\mathbf{x}}| \leq \frac{\rho}{2}\sigma$. From Lemma A.5 and Lemma A.1, $\left| \tilde{\Psi}(\mathbf{x}) - \tilde{\Psi}(\bar{\mathbf{x}}) \right| \leq (1 + 2\rho)(1 + 2\rho^2)|\mathbf{x} - \bar{\mathbf{x}}|$. We have that Ω contains at least half of the ball centered at $\Phi^{-1}(\mathbf{x})$ with radius $(\frac{\rho}{1+2\rho} - \frac{\rho(1+2\rho)(1+2\rho^2)}{2})\sigma \geq \frac{\rho}{5}\sigma$. This shows that $\tilde{\Phi}$ satisfies the condition (i). Similarly, the condition (ii) follows from (i) as $\tilde{\Phi}$ has bounded distortion (Lemma A.5) and geodesic distance is bounded by Euclidean distance (Lemma A.1). $\qquad \square$

B Proof of Lemma 7.1

Proof. We start with the evaluation of the x^i component of ∇v.

$$
\nabla^i v(\mathbf{x}) = \frac{C_t^2}{2tw_t^2(\mathbf{x})} \int_{\mathcal{M}} \int_{\mathcal{M}} \nabla^i x^j (x^j - y^j) R' \left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t} \right) R \left(\frac{|\mathbf{x}-\mathbf{y}'|^2}{4t} \right) u(\mathbf{y}) \mathrm{d}\mu'_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{y}}
$$
$$
- \frac{C_t^2}{2tw_t^2(\mathbf{x})} \int_{\mathcal{M}} \int_{\mathcal{M}} \nabla^i x^j (x^j - y'^j) R' \left(\frac{|\mathbf{x}-\mathbf{y}'|^2}{4t} \right) R \left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t} \right) u(\mathbf{y}) \mathrm{d}\mu'_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{y}}
$$
$$
= \frac{C_t^2}{4tw_t^2(\mathbf{x})} \int_{\mathcal{M}} \int_{\mathcal{M}} K^i(\mathbf{x}, \mathbf{y}, \mathbf{y}'; t)(u(\mathbf{y}) - u(\mathbf{y}')) \mathrm{d}\mu'_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{y}}
$$

where we set

$$
K^i(\mathbf{x}, \mathbf{y}, \mathbf{y}'; t) = \nabla^i x^j (x^j - y^j) R' \left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t} \right) R \left(\frac{|\mathbf{x}-\mathbf{y}'|^2}{4t} \right)
$$
$$
- \nabla^i x^j (x^j - y'^j) R' \left(\frac{|\mathbf{x}-\mathbf{y}'|^2}{4t} \right) R \left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t} \right).
$$

Think of $\nabla^i x^j$ as the i, j entry of the matrix $[\nabla^i x^j]$ and we have

$$
\begin{aligned}
\nabla^i x^j \nabla^l x^i &= (\partial_{i'} x^i) g^{i'j'} (\partial_{j'} x^j)(\partial_{s'} x^l) g^{s't'} (\partial_{t'} x^i) \\
&= g_{i't'} g^{i'j'} (\partial_{j'} x^j) g^{s't'} (\partial_{s'} x^l) \\
&= \delta_{j't'} (\partial_{j'} x^j) g^{s't'} (\partial_{s'} x^l) \\
&= (\partial_{j'} x^j) g^{s'j'} (\partial_{s'} x^l) \\
&= \nabla^l x^j.
\end{aligned}
$$

This shows that the matrix $[\nabla^i x^j]$ is idempotent. At the same time, $[\nabla^i x^j]$ is symmetric, which implies that the eigenvalues of $\nabla \mathbf{x}$ are either 1 or 0. Then we have the following upper bounds. There exists a constant C depending only on the maximum of R and R' so that

$$
\sum_{i=1}^{d} K^i(\mathbf{x}, \mathbf{y}, \mathbf{y}'; t)^2
$$
$$
\leq 2 \left(R' \left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t} \right) R \left(\frac{|\mathbf{x}-\mathbf{y}'|^2}{4t} \right) \right)^2 \| [\nabla^i x^j](\mathbf{x}-\mathbf{y}) \|^2
$$
$$
+ 2 \left(R' \left(\frac{|\mathbf{x}-\mathbf{y}'|^2}{4t} \right) R \left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t} \right) \right)^2 \| [\nabla^i x^j](\mathbf{x}-\mathbf{y}') \|^2
$$
$$
\leq C R' \left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t} \right) R \left(\frac{|\mathbf{x}-\mathbf{y}'|^2}{4t} \right) \| \mathbf{x}-\mathbf{y} \|^2
$$
$$
+ C R' \left(\frac{|\mathbf{x}-\mathbf{y}'|^2}{4t} \right) R \left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t} \right) \| \mathbf{x}-\mathbf{y}' \|^2
$$

There exists a constant C independent of t so that

$$
C_t^2 \int_{\mathcal{M}} \int_{\mathcal{M}} \frac{\sum_{i=1}^{d} K^i(\mathbf{x}, \mathbf{y}, \mathbf{y}'; t)^2}{t} \mathrm{d}\mu_{\mathbf{y}} \mathrm{d}\mu'_{\mathbf{y}}
$$
$$
\leq C \int_{\mathcal{M}} C_t R' \left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t} \right) \frac{\| \mathbf{x}-\mathbf{y} \|^2}{t} \mathrm{d}\mu_{\mathbf{y}} \int_{\mathcal{M}} C_t R \left(\frac{|\mathbf{x}-\mathbf{y}'|^2}{4t} \right) \mathrm{d}\mu'_{\mathbf{y}}
$$
$$
+ C \int_{\mathcal{M}} C_t R' \left(\frac{|\mathbf{x}-\mathbf{y}'|^2}{4t} \right) \frac{\| \mathbf{x}-\mathbf{y}' \|^2}{t} \mathrm{d}\mu'_{\mathbf{y}} \int_{\mathcal{M}} C_t R \left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t} \right) \mathrm{d}\mu_{\mathbf{y}}
$$
$$
\leq C
$$

Since R has a compact support, only when $|\mathbf{y}-\mathbf{y}'|^2 < 16t$ and $|\mathbf{x}-\frac{\mathbf{y}+\mathbf{y}'}{2}|^2 < 4t$ is $K^i(\mathbf{x},\mathbf{y},\mathbf{y}';t) \neq 0$. Thus from the assumption on R, we have

$$K^i(\mathbf{x},\mathbf{y},\mathbf{y}';t)^2 \leq \frac{1}{\delta_0^2}K^i(\mathbf{x},\mathbf{y},\mathbf{y}';t)^2 R\left(\frac{|\mathbf{y}-\mathbf{y}'|^2}{32t}\right) R\left(\frac{|\mathbf{x}-\frac{\mathbf{y}+\mathbf{y}'}{2}|^2}{8t}\right).$$

We can upper bound the norm of ∇v as follows:

$$|\nabla v(\mathbf{x})|^2 = \frac{C_t^4}{16t^2 w_t^4(\mathbf{x})}\sum_{i=1}^d \left(\int_\mathcal{M}\int_\mathcal{M} K^i(\mathbf{x},\mathbf{y},\mathbf{y}';t)(u(\mathbf{y})-u(\mathbf{y}'))\mathrm{d}\mathbf{y}'\mathrm{d}\mathbf{y}\right)^2$$

$$\leq \frac{C_t^4}{16t^2 w_t^4(\mathbf{x})}\sum_{i=1}^k \int_\mathcal{M}\int_\mathcal{M} K_i^2(\mathbf{x},\mathbf{y},\mathbf{y}';t)\left(R\left(\frac{|\mathbf{y}-\mathbf{y}'|^2}{32t}\right)R\left(\frac{|\mathbf{x}-\frac{\mathbf{y}+\mathbf{y}'}{2}|^2}{8t}\right)\right)^{-1}\mathrm{d}\mu_\mathbf{y}'\mathrm{d}\mu_\mathbf{y}$$

$$\int_\mathcal{M}\int_\mathcal{M} R\left(\frac{|\mathbf{x}-\frac{\mathbf{y}+\mathbf{y}'}{2}|^2}{8t}\right)R\left(\frac{|\mathbf{y}-\mathbf{y}'|^2}{32t}\right)(u(\mathbf{y})-u(\mathbf{y}'))^2\mathrm{d}\mu_\mathbf{y}'\mathrm{d}\mu_\mathbf{y}$$

$$= \frac{C_t^4}{16t\delta_0^2 w_t^4(\mathbf{x})}\int_\mathcal{M}\int_\mathcal{M}\frac{\sum_{i=1}^d K^i(\mathbf{x},\mathbf{y},\mathbf{y}';t)^2}{t}\mathrm{d}\mu_\mathbf{y}'\mathrm{d}\mu_\mathbf{y}$$

$$\int_\mathcal{M}\int_\mathcal{M} R\left(\frac{|\mathbf{x}-\frac{\mathbf{y}+\mathbf{y}'}{2}|^2}{8t}\right)R\left(\frac{|\mathbf{y}-\mathbf{y}'|^2}{32t}\right)(u(\mathbf{y})-u(\mathbf{y}'))^2\mathrm{d}\mu_\mathbf{y}'\mathrm{d}\mu_\mathbf{y}$$

$$\leq \frac{CC_t^2}{t}\int_\mathcal{M}\int_\mathcal{M} R\left(\frac{|\mathbf{x}-\frac{\mathbf{y}+\mathbf{y}'}{2}|^2}{8t}\right)R\left(\frac{|\mathbf{y}-\mathbf{y}'|^2}{32t}\right)(u(\mathbf{y})-u(\mathbf{y}'))^2\mathrm{d}\mu_\mathbf{y}'\mathrm{d}\mu_\mathbf{y}.$$

Finally, we have

$$\int_\mathcal{M}|\nabla v(\mathbf{x})|^2\mathrm{d}\mu_\mathbf{x}$$

$$\leq \frac{CC_t^2}{t}\int_\mathcal{M}\left(\int_\mathcal{M}\int_\mathcal{M} R\left(\frac{|\mathbf{x}-\frac{\mathbf{y}+\mathbf{y}'}{2}|^2}{8t}\right)R\left(\frac{|\mathbf{y}-\mathbf{y}'|^2}{32t}\right)(u(\mathbf{y})-u(\mathbf{y}'))^2\mathrm{d}\mu_\mathbf{y}'\mathrm{d}\mu_\mathbf{y}\right)\mathrm{d}\mu_\mathbf{x}$$

$$= \frac{CC_t^2}{t}\int_\mathcal{M}\int_\mathcal{M}\left(\int_\mathcal{M} R\left(\frac{|\mathbf{x}-\frac{\mathbf{y}+\mathbf{y}'}{2}|^2}{8t}\right)\mathrm{d}\mu_\mathbf{x}\right)R\left(\frac{|\mathbf{y}-\mathbf{y}'|^2}{32t}\right)(u(\mathbf{y})-u(\mathbf{y}'))^2\mathrm{d}\mu_\mathbf{y}'\mathrm{d}\mu_\mathbf{y}$$

$$\leq \frac{CC_t}{t}\int_\mathcal{M}\int_\mathcal{M} R\left(\frac{|\mathbf{y}-\mathbf{y}'|^2}{32t}\right)(u(\mathbf{y})-u(\mathbf{y}'))^2\mathrm{d}\mu_\mathbf{y}'\mathrm{d}\mu_\mathbf{y}.$$

This proves the Lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

C Proof of Lemma 7.2

Based on the partition and the parametrization of the manifold $\mathcal{M}$ introduced in Section 5, we have

$$\int_\mathcal{M}\int_\mathcal{M} R\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{32t}\right)(u(\mathbf{x})-u(\mathbf{y}))^2\mathrm{d}\mu_\mathbf{x}\mathrm{d}\mu_\mathbf{y} \tag{C.1}$$

$$= \sum_{i=1}^N\int_\mathcal{M}\int_{\mathcal{O}_i} R\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{32t}\right)(u(\mathbf{x})-u(\mathbf{y}))^2\mathrm{d}\mu_\mathbf{x}\mathrm{d}\mu_\mathbf{y}$$

$$= \sum_{i=1}^N\int_{B_{\mathbf{q}_i}^{2\delta}}\int_{\mathcal{O}_i} R\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{32t}\right)(u(\mathbf{x})-u(\mathbf{y}))^2\mathrm{d}\mu_\mathbf{x}\mathrm{d}\mu_\mathbf{y}.$$

29

For any $\mathbf{x} \in \mathcal{O}_i$ and $\mathbf{y} \in B_{\mathbf{q}_i}^{2\delta}$, let

$$\mathbf{z}_j = \Phi_i \left(\left(\frac{j}{16} \right) \Phi_i^{-1}(\mathbf{x}) + \left(1 - \frac{j}{16} \right) \Phi_i^{-1}(\mathbf{y}) \right), \quad j = 0, \cdots, 16. \tag{C.2}$$

Apparently, $\mathbf{z}_0 = \mathbf{x}$, $\mathbf{z}_{16} = \mathbf{y}$. Since Ω_i is convex, we have $\Phi_i^{-1}(\mathbf{z}_j) \in \Omega_i$, $i = 0, \cdots, 16$. Then utilizing *locally small deformation* property of the parametrization, we obtain

$$\begin{aligned}
\|\mathbf{z}_j - \mathbf{z}_{j+1}\| &\leq 2\|\Phi^{-1}(\mathbf{z}_j) - \Phi^{-1}(\mathbf{z}_{j+1})\| \\
&\leq \frac{1}{8}\|\Phi^{-1}(\mathbf{x}) - \Phi^{-1}(\mathbf{y})\| \\
&\leq \frac{1}{4}\|\mathbf{x} - \mathbf{y}\|.
\end{aligned}$$

Now, we are ready to estimate the integrals in (C.1).

$$\begin{aligned}
&\int_{B_{\mathbf{q}_i}^{2\delta}} \int_{\mathcal{O}_i} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{32t} \right) (u(\mathbf{x}) - u(\mathbf{y}))^2 \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\mu_{\mathbf{y}} \\
\leq\ & 16 \sum_{j=0}^{15} \int_{B_{\mathbf{q}_i}^{2\delta}} \int_{\mathcal{O}_i} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{32t} \right) (u(\mathbf{z}_j) - u(\mathbf{z}_{j+1}))^2 \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\mu_{\mathbf{y}} \\
=\ & 16 \sum_{j=0}^{15} \int_{\mathcal{O}_i} \left[\int_{\mathcal{M}_{\mathbf{x}}^t} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{32t} \right) (u(\mathbf{z}_j) - u(\mathbf{z}_{j+1}))^2 \mathrm{d}\mu_{\mathbf{y}} \right] \mathrm{d}\mu_{\mathbf{x}}.
\end{aligned}$$

For any $\mathbf{y} \in \mathcal{M}_{\mathbf{x}}^t$,

$$\|\mathbf{z}_j - \mathbf{z}_{j+1}\|^2 \leq \frac{1}{16}\|\mathbf{x} - \mathbf{y}\|^2 \leq 2t, \quad j = 0, \cdots, 15, \tag{C.3}$$

which implies that

$$R\left(\frac{|\mathbf{z}_j - \mathbf{z}_{j+1}|^2}{4t} \right) \geq \delta_0, \quad j = 0, \cdots, 15. \tag{C.4}$$

Now, we have

$$\begin{aligned}
&\int_{\mathcal{O}_i} \left[\int_{\mathcal{M}_{\mathbf{x}}^t} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{32t} \right) (u(\mathbf{z}_j) - u(\mathbf{z}_{j+1}))^2 \mathrm{d}\mu_{\mathbf{y}} \right] \mathrm{d}\mu_{\mathbf{x}} \\
=\ & \int_{\mathcal{O}_i} \left[\int_{\mathcal{M}_{\mathbf{x}}^t} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{32t} \right) \left(R\left(\frac{|\mathbf{z}_j - \mathbf{z}_{j+1}|^2}{4t} \right) \right)^{-1} R\left(\frac{|\mathbf{z}_j - \mathbf{z}_{j+1}|^2}{4t} \right) (u(\mathbf{z}_j) - u(\mathbf{z}_{j+1}))^2 \mathrm{d}\mu_{\mathbf{y}} \right] \mathrm{d}\mu_{\mathbf{x}} \\
\leq\ & \frac{1}{\delta_0} \int_{\mathcal{O}_i} \left[\int_{\mathcal{M}_{\mathbf{x}}^t} R\left(\frac{|\mathbf{z}_j - \mathbf{z}_{j+1}|^2}{4t} \right) (u(\mathbf{z}_j) - u(\mathbf{z}_{j+1}))^2 \mathrm{d}\mu_{\mathbf{y}} \right] \mathrm{d}\mu_{\mathbf{x}} \\
=\ & \frac{1}{\delta_0} \int_{\Phi_i^{-1}(\mathcal{O}_i)} \left[\int_{\Phi_i^{-1}(\mathcal{M}_{\mathbf{x}}^t)} R\left(\frac{|\mathbf{z}_j - \mathbf{z}_{j+1}|^2}{4t} \right) (u(\mathbf{z}_j) - u(\mathbf{z}_{j+1}))^2 |\nabla\Phi(\theta_{\mathbf{y}})| \mathrm{d}\theta_{\mathbf{y}} \right] |\nabla\Phi(\theta_{\mathbf{x}})| \mathrm{d}\theta_{\mathbf{x}} \\
\leq\ & \frac{4}{\delta_0} \int_{\Phi_i^{-1}(\mathcal{O}_i)} \left[\int_{\Phi_i^{-1}(\mathcal{M}_{\mathbf{x}}^t)} R\left(\frac{|\mathbf{z}_j - \mathbf{z}_{j+1}|^2}{4t} \right) (u(\mathbf{z}_j) - u(\mathbf{z}_{j+1}))^2 \mathrm{d}\theta_{\mathbf{y}} \right] \mathrm{d}\theta_{\mathbf{x}}
\end{aligned}$$

where $\theta_{\mathbf{x}} = \Phi_i^{-1}(\mathbf{x})$, $\theta_{\mathbf{y}} = \Phi_i^{-1}(\mathbf{y})$.

Let

$$\theta_{\mathbf{z}_j} = \Phi_i^{-1}(\mathbf{z}_j) = \frac{j}{16}\theta_{\mathbf{x}} + \left(1 - \frac{j}{16}\right)\theta_{\mathbf{y}}, \quad j = 0, \cdots, 16. \tag{C.5}$$

It is easy to show that $\Phi_i(\theta_{\mathbf{z}_j}) = \mathbf{z}_j \in B_{\mathbf{q}_i}^{2\delta}$, $j = 0, \cdots, 16$ by using the facts that for any $\mathbf{y} \in \mathcal{M}_{\mathbf{x}}^t$

$$\|\mathbf{z}_j - \mathbf{x}\| \le \sum_{l=1}^{j} \|\mathbf{z}_l - \mathbf{z}_{l-1}\| \le \frac{j}{4}\|\mathbf{x} - \mathbf{y}\| \le 15\sqrt{2t}, \quad j = 1, \cdots, 15, \tag{C.6}$$

and $\mathbf{x} \in B_{\mathbf{q}_i}^{\delta}$ and $15\sqrt{2t} \le r$. Then we have

$$\theta_{\mathbf{z}_j} \in \Phi_i^{-1}\left(B_{\mathbf{q}_i}^{2\delta}\right), \quad j = 0, \cdots, 16. \tag{C.7}$$

By changing variable, we obtain

$$\int_{\Phi_i^{-1}(\mathcal{O}_i)} \left[\int_{\Phi_i^{-1}(\mathcal{M}_{\mathbf{x}}^t)} R\left(\frac{|\mathbf{z}_j - \mathbf{z}_{j+1}|^2}{4t}\right)(u(\mathbf{z}_j) - u(\mathbf{z}_{j+1}))^2 d\theta_{\mathbf{y}}\right] d\theta_{\mathbf{x}}$$

$$\le 8^k \int_{\Phi_i^{-1}(B_{\mathbf{q}_i}^{2\delta})} \int_{\Phi_i^{-1}(B_{\mathbf{q}_i}^{2\delta})} R\left(\frac{|\mathbf{z}_j - \mathbf{z}_{j+1}|^2}{4t}\right)(u(\mathbf{z}_j) - u(\mathbf{z}_{j+1}))^2 d\theta_{\mathbf{z}_j} d\theta_{\mathbf{z}_{j+1}}$$

$$\le 4 \cdot 8^k \int_{\Phi_i^{-1}(B_{\mathbf{q}_i}^{2\delta})} \int_{\Phi_i^{-1}(B_{\mathbf{q}_i}^{2\delta})} R\left(\frac{|\mathbf{z}_j - \mathbf{z}_{j+1}|^2}{4t}\right)(u(\mathbf{z}_j) - u(\mathbf{z}_{j+1}))^2 \left|\nabla\Phi(\theta_{\mathbf{z}_j})\right| \left|\nabla\Phi(\theta_{\mathbf{z}_{j+1}})\right| d\theta_{\mathbf{z}_j} d\theta_{\mathbf{z}_{j+1}}$$

$$= 4 \cdot 8^k \int_{B_{\mathbf{q}_i}^{2\delta}} \int_{B_{\mathbf{q}_i}^{2\delta}} R\left(\frac{|\mathbf{z}_j - \mathbf{z}_{j+1}|^2}{4t}\right)(u(\mathbf{z}_j) - u(\mathbf{z}_{j+1}))^2 d\mu_{\mathbf{z}_j} d\mu_{\mathbf{z}_{j+1}}$$

$$\le C \int_{\mathcal{M}} \int_{B_{\mathbf{q}_i}^{2\delta}} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t}\right)(u(\mathbf{x}) - u(\mathbf{y}))^2 d\mu_{\mathbf{x}} d\mu_{\mathbf{y}}$$

Finally, we can prove the lemma as follows.

$$\int_{\mathcal{M}} \int_{\mathcal{M}} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{32t}\right)(u(\mathbf{x}) - u(\mathbf{y}))^2 d\mu_{\mathbf{x}} d\mu_{\mathbf{y}}$$

$$\le C \sum_{i=1}^{N} \int_{\mathcal{M}} \int_{B_{\mathbf{q}_i}^{2\delta}} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t}\right)(u(\mathbf{x}) - u(\mathbf{y}))^2 d\mu_{\mathbf{x}} d\mu_{\mathbf{y}}$$

$$\le CN \int_{\mathcal{M}} \int_{\mathcal{M}} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t}\right)(u(\mathbf{x}) - u(\mathbf{y}))^2 d\mu_{\mathbf{x}} d\mu_{\mathbf{y}}.$$

D Proof of Theorem 6.1

First, we introduce a smooth function u_t that approximates $\mathbf{u}$ at the samples P.

$$u_t(\mathbf{x}) = \frac{C_t}{w_{t,h}(\mathbf{x})} \sum_{i=1}^{n} R\left(\frac{|\mathbf{x} - \mathbf{p}_i|^2}{4t}\right) u_i V_i, \tag{D.1}$$

where $w_{t,h}(\mathbf{x}) = C_t \sum_{i=1}^{n} R\left(\frac{|\mathbf{x} - \mathbf{p}_i|^2}{4t}\right) V_i$. We have the following lemma about the function $w_{t,h}$.

Lemma D.1. *Assume the submanifold $\mathcal{M}$ and $\partial\mathcal{M}$ are C^2 smooth and t, $\frac{h(P,\mathbf{V},\mathcal{M})}{t^{1/2}}$ are sufficiently small. There exists a constant C_1, C_2 and C, so that*

$$C_1 \le w_{t,h}(\mathbf{x}) \le C_2, \text{ and } |\nabla w_{t,h}(\mathbf{x})| \le \frac{C}{t^{1/2}}$$

Proof. Using the definition of $h(P, \mathbf{V}, \mathcal{M})$,

$$\left| w_{t,h}(\mathbf{x}) - C_t \int_{\mathcal{M}} R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t} \right) \mathrm{d}\mu_{\mathbf{y}} \right| \leq \frac{Ch(P, \mathbf{V}, \mathcal{M})}{t^{1/2}},$$

which shows the bounds on $w_{t,h}(\mathbf{x})$. Next, we show the bound on the gradient.

$$
\begin{aligned}
|\nabla w_{t,h}(\mathbf{x})|^2 &\leq \sum_{i=1}^{d} \left(\frac{\partial w_{t,h}}{\partial x^i} \right)^2 = \sum_{i=1}^{d} \left(\sum_{j=1}^{n} C_t R'\left(\frac{|\mathbf{x} - \mathbf{p}_j|^2}{4t} \right) \frac{x^i - p_j^i}{2t} V_j \right)^2 \\
&\leq \left(\sum_{j=1}^{n} C_t R'\left(\frac{|\mathbf{x} - \mathbf{p}_j|^2}{4t} \right) \frac{|\mathbf{x} - \mathbf{p}_j|^2}{4t^2} V_j \right) \left(\sum_{j=1}^{n} C_t R'\left(\frac{|\mathbf{x} - \mathbf{p}_j|^2}{4t} \right) V_j \right) \\
&\leq \frac{C}{t}.
\end{aligned}
$$

$\square$

Now we are ready to give the proof of Theorem 6.1.

Proof. In the definition of u_t and $w_{t,h}$ in (D.1), replace t with $t' = t/18$. We have

$$
\begin{aligned}
&\int_{\mathcal{M}} \int_{\mathcal{M}} R_{t'}(\mathbf{x}, \mathbf{y}) \left(u_t(\mathbf{x}) - u_t(\mathbf{y}) \right)^2 \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\mu_{\mathbf{y}} \\
&= \int_{\mathcal{M}} \int_{\mathcal{M}} R_{t'}(\mathbf{x}, \mathbf{y}) \left(\frac{1}{w_{t',h}(\mathbf{x})} \sum_{i=1}^{n} R_{t'}(\mathbf{x}, \mathbf{p}_i) u_i V_i - \frac{1}{w_{t',h}(\mathbf{y})} \sum_{j=1}^{n} R_{t'}(\mathbf{p}_j, \mathbf{y}) u_j V_j \right)^2 \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\mu_{\mathbf{y}} \\
&= \int_{\mathcal{M}} \int_{\mathcal{M}} R_{t'}(\mathbf{x}, \mathbf{y}) \left(\frac{1}{w_{t',h}(\mathbf{x}) w_{t',h}(\mathbf{y})} \sum_{i,j=1}^{n} R_{t'}(\mathbf{x}, \mathbf{p}_i) R_{t'}(\mathbf{p}_j, \mathbf{y}) V_i V_j (u_i - u_j) \right)^2 \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\mu_{\mathbf{y}} \\
&\leq \int_{\mathcal{M}} \int_{\mathcal{M}} R_{t'}(\mathbf{x}, \mathbf{y}) \frac{1}{w_{t',h}(\mathbf{x}) w_{t',h}(\mathbf{y})} \sum_{i,j=1}^{n} R_{t'}(\mathbf{x}, \mathbf{p}_i) R_{t'}(\mathbf{p}_j, \mathbf{y}) V_i V_j (u_i - u_j)^2 \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\mu_{\mathbf{y}} \\
&= \sum_{i,j=1}^{n} \left(\int_{\mathcal{M}} \int_{\mathcal{M}} \frac{1}{w_{t',h}(\mathbf{x}) w_{t',h}(\mathbf{y})} R_{t'}(\mathbf{x}, \mathbf{p}_i) R_{t'}(\mathbf{p}_j, \mathbf{y}) R_{t'}(\mathbf{x}, \mathbf{y}) \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\mu_{\mathbf{y}} \right) V_i V_j (u_i - u_j)^2
\end{aligned}
$$

Denote

$$A = \int_{\mathcal{M}} \int_{\mathcal{M}} \frac{1}{w_{t',h}(\mathbf{x}) w_{t',h}(\mathbf{y})} R_{t'}(\mathbf{x}, \mathbf{p}_i) R_{t'}(\mathbf{p}_j, \mathbf{y}) R_{t'}(\mathbf{x}, \mathbf{y}) \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\mu_{\mathbf{y}}$$

and then notice only when $|\mathbf{p}_i - \mathbf{p}_j|^2 \leq 36t'$ is $A \neq 0$. For $|\mathbf{p}_i - \mathbf{p}_j|^2 \leq 36t'$, we have

$$
\begin{aligned}
A &\leq \int_{\mathcal{M}} \int_{\mathcal{M}} R_{t'}(\mathbf{x}, \mathbf{p}_i) R_{t'}(\mathbf{p}_j, \mathbf{y}) R_{t'}(\mathbf{x}, \mathbf{y}) R\left(\frac{|\mathbf{p}_i - \mathbf{p}_j|^2}{72t'} \right)^{-1} R\left(\frac{|\mathbf{p}_i - \mathbf{p}_j|^2}{72t'} \right) \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\mu_{\mathbf{y}} \\
&\leq \frac{CC_t}{\delta_0} \int_{\mathcal{M}} \int_{\mathcal{M}} R_{t'}(\mathbf{x}, \mathbf{p}_i) R_{t'}(\mathbf{p}_j, \mathbf{y}) R\left(\frac{|\mathbf{p}_i - \mathbf{p}_j|^2}{72t'} \right) \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\mu_{\mathbf{y}} \\
&\leq CC_t \int_{\mathcal{M}} \int_{\mathcal{M}} R_{t'}(\mathbf{x}, \mathbf{p}_i) R_{t'}(\mathbf{p}_j, \mathbf{y}) R\left(\frac{|\mathbf{p}_i - \mathbf{p}_j|^2}{72t'} \right) \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\mu_{\mathbf{y}} \\
&\leq CC_t R\left(\frac{|\mathbf{p}_i - \mathbf{p}_j|^2}{4t} \right).
\end{aligned}
$$

Combining the above two inequalities and using Lemma 7.2, we obtain

$$C\frac{C_t}{t}\sum_{i,j=1}^{n}R\left(\frac{|\mathbf{p}_i-\mathbf{p}_j|^2}{4t}\right)(u_i-u_j)^2V_iV_j\geq\int_{\mathcal{M}}(u_t(\mathbf{x})-\bar{u}_t)^2\mathrm{d}\mu_{\mathbf{x}}\tag{D.2}$$

We now lower bound the RHS of the above equation.

$$|\mathcal{M}||\bar{u}_t|\;=\;\left|\int_{\mathcal{M}}u_t(\mathbf{x})\mathrm{d}\mu_{\mathbf{x}}\right|=\left|\sum_{j=1}^{n}\left(u_jV_j\int_{\mathcal{M}}\frac{C'_t}{w_{t',h}(\mathbf{x})}R\left(\frac{|\mathbf{x}-\mathbf{p}_j|^2}{4t'}\right)\mathrm{d}\mu_{\mathbf{x}}\right)\right|.$$

Let $q(\mathbf{x})=\frac{C'_t}{w_{t',h}(\mathbf{x})}R\left(\frac{|\mathbf{x}-\mathbf{p}_j|^2}{4t'}\right)$. There exists a constant C so that $|q(\mathbf{x})|\leq CC'_t$ and

$$|\nabla q(\mathbf{x})|\leq\frac{C'_t}{w_{t',h}(\mathbf{x})}\left|\nabla R\left(\frac{|\mathbf{x}-\mathbf{p}_j|^2}{4t'}\right)\right|+\frac{C'_t|\nabla w_{t',h}(\mathbf{x})|}{w_{t,h}^2(\mathbf{x})}R\left(\frac{|\mathbf{x}-\mathbf{p}_j|^2}{4t'}\right)\leq\frac{CC'_t}{t^{1/2}}.$$

Then, using the definition of the integral accuracy index, there exists a constant C

$$\left|\int_{\mathcal{M}}\frac{C'_t}{w_{t',h}(\mathbf{x})}R\left(\frac{|\mathbf{x}-\mathbf{p}_j|^2}{4t'}\right)\mathrm{d}\mu_{\mathbf{x}}-\sum_{i=1}^{n}\frac{C'_t}{w_{t',h}(\mathbf{p}_i)}R\left(\frac{|\mathbf{p}_i-\mathbf{p}_j|^2}{4t'}\right)V_i\right|\leq\frac{Ch}{t^{1/2}}.$$

Thus we have

$$|\mathcal{M}||\bar{u}_t|\tag{D.3}$$

$$\leq\left|\sum_{i,j=1}^{n}\frac{C'_t}{w_{t',h}(\mathbf{p}_i)}R\left(\frac{|\mathbf{p}_i-\mathbf{p}_j|^2}{4t'}\right)u_jV_iV_j\right|+\frac{Ch}{t^{1/2}}(\sum_{j=1}^{n}|u_jV_j|)$$

$$\leq\left|\sum_{i=1}^{n}u_t(\mathbf{p}_i)V_i\right|+\frac{Ch}{t^{1/2}}(\sum_{i=1}^{n}|u_iV_i|)$$

$$=\frac{1}{|\mathcal{M}|}\left|\sum_{i,j=1}^{n}\frac{C'_t}{w_{t',h}(\mathbf{p}_i)}R\left(\frac{|\mathbf{p}_i-\mathbf{p}_j|^2}{4t'}\right)(u_j-u_i)V_iV_j\right|+\frac{Ch}{t^{1/2}}(\sum_{i=1}^{n}u_i^2V_i)^{1/2}$$

$$\leq\frac{CC_t^{1/2}}{\sqrt{|\mathcal{M}|}}\left(\sum_{i,j=1}^{n}R\left(\frac{|\mathbf{x}-\mathbf{p}_j|^2}{4t'}\right)(u_i-u_j)^2V_iV_j\right)^{1/2}+\frac{Ch}{t^{1/2}}(\sum_{i=1}^{n}u_i^2V_i)^{1/2}$$

where the first equality is due to $\sum_{i=1}^{n}u_iV_i=0$. Denote

$$A=\int_{\mathcal{M}}\frac{C_t}{w_{t',h}^2(\mathbf{x})}R\left(\frac{|\mathbf{x}-\mathbf{p}_i|^2}{4t'}\right)R\left(\frac{|\mathbf{x}-\mathbf{p}_l|^2}{4t'}\right)\mathrm{d}\mu_{\mathbf{x}}-$$

$$\sum_{j=1}^{n}\frac{C_t}{w_{t',h}^2(\mathbf{p}_j)}R\left(\frac{|\mathbf{p}_j-\mathbf{p}_i|^2}{4t'}\right)R\left(\frac{|\mathbf{p}_j-\mathbf{p}_l|^2}{4t'}\right)V_j$$

and then $|A|\leq\frac{Ch}{t^{1/2}}$. At the same time, notice that only when $|\mathbf{p}_i-\mathbf{p}_l|^2<16t'$ is $A\neq0$. Thus we have

$$|A|\leq\frac{1}{\delta_0}|A|R(\frac{|\mathbf{p}_i-\mathbf{p}_l|^2}{32t'}),$$

33

and

$$\left| \int_{\mathcal{M}} u_t^2(\mathbf{x}) \mathrm{d}\mu_{\mathbf{x}} - \sum_{j=1}^n u_t^2(\mathbf{p}_j) V_j \right| \tag{D.4}$$

$$\leq \sum_{i,l=1}^n |C_t u_i u_l V_i V_l| |A|$$

$$\leq \frac{Ch}{t^{1/2}} \sum_{i,l=1}^n \left| C_t R \left(\frac{|\mathbf{p}_i - \mathbf{p}_l|^2}{32t'} \right) u_i u_l V_i V_l \right|$$

$$\leq \frac{Ch}{t^{1/2}} \sum_{i,l=1}^n C_t R \left(\frac{|\mathbf{p}_i - \mathbf{p}_l|^2}{32t'} \right) u_i^2 V_i V_l \leq \frac{Ch}{t^{1/2}} \sum_{i=1}^n u_i^2 V_i.$$

Now combining Equation (D.2), (D.3) and (D.4), we have for small t

$$\sum_{i=1}^n u_t^2(\mathbf{p}_i) V_i = \int_{\mathcal{M}} u_t^2(\mathbf{x}) \mathrm{d}\mu_{\mathbf{x}} + \frac{Ch}{t^{1/2}} \sum_{i=1}^n u_i^2 V_i$$

$$\leq 2 \int_{\mathcal{M}} (u_t(\mathbf{x}) - \bar{u}_t)^2 \mathrm{d}\mu_{\mathbf{x}} + 2\bar{u}_t^2 |\mathcal{M}| + \frac{Ch}{t^{1/2}} \sum_{i=1}^n u_i^2 V_i$$

$$\leq \frac{CC_t}{t} \sum_{i,j=1}^n R \left(\frac{|\mathbf{p}_i - \mathbf{p}_j|^2}{4t} \right) (u_i - u_j)^2 V_i V_j + \frac{Ch}{t} \sum_{i=1}^n u_i^2 V_i.$$

Here we use the fact that $t = 18t'$ hence

$$R \left(\frac{|\mathbf{p}_i - \mathbf{p}_j|^2}{4t'} \right) \leq \frac{1}{\delta_0} R \left(\frac{|\mathbf{p}_i - \mathbf{p}_j|^2}{4t} \right).$$

Let $\delta = \frac{w_{\min}}{2w_{\max} + w_{\min}}$ with $w_{\min} = \min_{\mathbf{x}} w_{t,h}(\mathbf{x})$ and $w_{\max} = \max_{\mathbf{x}} w_{t,h}(\mathbf{x})$. If

$$\sum_{i=1}^n u^2(\mathbf{p}_i) V_i \geq \delta^2 \sum_{i=1}^n u_i^2 V_i,$$

we have completed the proof. Otherwise, we have

$$\sum_{i=1}^n (u_i - u_t(\mathbf{p}_i))^2 V_i = \sum_{i=1}^n u_i^2 V_i + \sum_{i=1}^n u_t(\mathbf{p}_i)^2 V_i - 2 \sum_{i=1}^n u_i u_t(\mathbf{p}_i) V_i \geq (1 - \delta)^2 \sum_{i=1}^n u_i^2 V_i.$$

This enables us to prove the theorem as follows.

$$C_t \sum_{i,j=1}^n R \left(\frac{|\mathbf{p}_i - \mathbf{p}_j|^2}{4t'} \right) (u_i - u_j)^2 V_i V_j = 2C_t \sum_{i,j=1}^n R \left(\frac{|\mathbf{p}_i - \mathbf{p}_j|^2}{4t'} \right) u_i (u_i - u_j) V_i V_j$$

$$= 2 \sum_{i=1}^n (u_i - u_t(\mathbf{p}_i))^2 w_{t,h}(\mathbf{p}_i) V_i + 2 \sum_{i=1}^n u_t(\mathbf{p}_i)(u_i - u(\mathbf{p}_i)) w_{t,h}(\mathbf{p}_i) V_i$$

$$\geq 2 \sum_{i=1}^n (u_i - u_t(\mathbf{p}_i))^2 w_{t,h}(\mathbf{p}_i) V_i - 2 \left(\sum_{i=1}^n u_t^2(\mathbf{p}_i) w_{t,h}(\mathbf{p}_i) V_i \right)^{1/2} \left(\sum_{i=1}^n (u_i - u_t(\mathbf{p}_i))^2 w_{t,h}(\mathbf{p}_i) V_i \right)^{1/2}$$

$$\geq 2 w_{\min} \sum_{i=1}^n (u_i - u_t(\mathbf{p}_i))^2 V_i - 2 w_{\max} \left(\sum_{i=1}^n u_t^2(\mathbf{p}_i) V_i \right)^{1/2} \left(\sum_{i=1}^n (u_i - u_t(\mathbf{p}_i))^2 V_i \right)^{1/2}$$

$$\geq 2 (w_{\min}(1 - \delta)^2 - w_{\max}\delta(1 - \delta)) \sum_{i=1}^n u_i^2 V_i \geq w_{\min}(1 - \delta)^2 \sum_{i=1}^n u_i^2 V_i.$$

34

□

E   Estimation of $\|\nabla L_t(u_t - u_{t,h})\|_{L^2(\mathcal{M})}$

In this section, we upper bound $\nabla L_t(u_t - u_{t,h})\|_{L^2(\mathcal{M})}$. Remember that u_t satisfies the integral equation (4.2) and

$$u_{t,h}(\mathbf{x}) = \frac{1}{w_{t,h}(\mathbf{x})} \left(\sum_{\mathbf{p}_j \in P} R_t(\mathbf{x}, \mathbf{p}_j) u_j V_j - t \sum_{\mathbf{p}_j \in P} \bar{R}_t(\mathbf{x}, \mathbf{p}_j) f_j V_j \right),$$

where $\mathbf{u} = (u_1, \cdots, u_n)^t$ with $\sum_{i=1}^n u_i V_i = 0$ solves the problem (3.2), $f_j = f(\mathbf{p}_j)$ and $w_{t,h}(\mathbf{x}) = \sum_{\mathbf{p}_j \in P} R_t(\mathbf{x}, \mathbf{p}_j) V_j$.

$\nabla L_t(u_t - u_{t,h})\|_{L^2(\mathcal{M})}$ is splitted to two terms,

$$\nabla L_t(u_t - u_{t,h}) = \nabla(L_t u_t - L_{t,h} u_{t,h}) + \nabla(L_{t,h} u_{t,h} - L_t u_{t,h}).$$

The second term is easy to bound.

$$\|\nabla (L_t(u_t) - L_{t,h}(u_{t,h}))\|_{L^2(\mathcal{M})} \tag{E.1}$$

$$= \left(\int_{\mathcal{M}} \left(\int_{\mathcal{M}} \nabla_{\mathbf{x}} \bar{R}_t(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) - \sum_{\mathbf{p}_j \in P} \nabla_{\mathbf{x}} \bar{R}_t(\mathbf{x}, \mathbf{p}_j) f(\mathbf{p}_j) V_j \right)^2 \mathrm{d}\mu_{\mathbf{x}} \right)^{1/2}$$

$$\leq \frac{Ch}{t} \|f\|_{C^1(\mathcal{M})}.$$

The first term is further splited by defining

$$a_{t,h}(\mathbf{x}) = \frac{1}{w_{t,h}(\mathbf{x})} \sum_{\mathbf{p}_j \in P} R_t(\mathbf{x}, \mathbf{p}_j) u_j V_j,$$

$$c_{t,h}(\mathbf{x}) = -\frac{t}{w_{t,h}(\mathbf{x})} \sum_{\mathbf{p}_j \in P} \bar{R}_t(\mathbf{x}, \mathbf{p}_j) f(\mathbf{p}_j) V_j,$$

To simplify the notation, we denote $h = h(P, \mathbf{V}, \mathcal{M})$ and $n = |P|$. Consider $\|\nabla(L_t a_{t,h} - L_{t,h} a_{t,h})\|_{L_2}$.

$$\int_{\mathcal{M}} |\nabla a_{t,h}(\mathbf{x})|^2 \left| \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} - \sum_{\mathbf{p}_j \in P} R_t(\mathbf{x}, \mathbf{p}_j) V_j \right|^2 \mathrm{d}\mu_{\mathbf{x}} \qquad \text{(E.2)}$$

$$\leq \frac{Ch^2}{t} \int_{\mathcal{M}} |\nabla a_{t,h}(\mathbf{x})|^2 \mathrm{d}\mu_{\mathbf{x}}$$

$$\leq \frac{Ch^2}{t} \left(\int_{\mathcal{M}} \left| \frac{1}{w_{t,h}(\mathbf{x})} \sum_{\mathbf{p}_j \in P} \nabla R_t(\mathbf{x}, \mathbf{p}_j) u_j V_j \right|^2 \mathrm{d}\mu_{\mathbf{x}} \right.$$

$$\left. + \int_{\mathcal{M}} \left| \frac{\nabla w_{t,h}(\mathbf{x})}{w_{t,h}^2(\mathbf{x})} \sum_{\mathbf{p}_j \in P} R_t(\mathbf{x}, \mathbf{p}_j) u_j V_j \right|^2 \mathrm{d}\mu_{\mathbf{x}} \right)$$

$$\leq \frac{Ch^2}{t^2} \int_{\mathcal{M}} \left| \sum_{\mathbf{p}_j \in P} R_{2t}(\mathbf{x}, \mathbf{p}_j) u_j V_j \right|^2 \mathrm{d}\mu_{\mathbf{x}}$$

$$\leq \frac{Ch^2}{t^2} \left(\sum_{j=1}^{n} u_j^2 V_j \int_{\mathcal{M}} R_{2t}(\mathbf{x}, \mathbf{p}_j) \mathrm{d}\mu_{\mathbf{x}} \right) \leq \frac{Ch^2}{t^2} \sum_{j=1}^{n} u_j^2 V_j.$$

where $R_{2t}(\mathbf{x}, \mathbf{p}_j) = C_t R\left(\frac{|\mathbf{x} - \mathbf{p}_j|^2}{8t} \right)$. Here we use the assumption that $R(s) > \delta_0$ for all $0 \leq s \leq 1/2$.

$$\int_{\mathcal{M}} |a_{t,h}(\mathbf{x})|^2 \left| \int_{\mathcal{M}} \nabla R_t(\mathbf{x}, \mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} - \sum_{\mathbf{p}_j \in P} \nabla R_t(\mathbf{x}, \mathbf{p}_j) V_j \right|^2 \mathrm{d}\mu_{\mathbf{x}} \qquad \text{(E.3)}$$

$$\leq \frac{Ch^2}{t^2} \int_{\mathcal{M}} |a_{t,h}(\mathbf{x})|^2 \mathrm{d}\mu_{\mathbf{x}} \leq \frac{Ch^2}{t^2} \sum_{j=1}^{n} u_j^2 V_j.$$

Let

$$B = C_t \int_{\mathcal{M}} \frac{1}{w_{t,h}(\mathbf{y})} \nabla R\left(\frac{|\mathbf{x} - \mathbf{y}|^2}{4t} \right) R\left(\frac{|\mathbf{p}_i - \mathbf{y}|^2}{4t} \right) \mathrm{d}\mu_{\mathbf{y}}$$

$$- C_t \sum_{\mathbf{p}_j \in P} \frac{1}{w_{t,h}(\mathbf{p}_j)} \nabla R\left(\frac{|\mathbf{x} - \mathbf{p}_j|^2}{4t} \right) R\left(\frac{|\mathbf{p}_i - \mathbf{p}_j|^2}{4t} \right) V_j.$$

We have $|B| < \frac{Ch}{t^{1/2}}$ for some constant C independent of t. In addition, notice that only when $|\mathbf{x} - \mathbf{x}_i|^2 \leq 16t$ is $B \neq 0$, which implies

$$|B| \leq \frac{1}{\delta_0} |B| R\left(\frac{|\mathbf{x} - \mathbf{p}_i|^2}{32t} \right).$$

Then we have

$$\int_{\mathcal{M}} \left| \int_{\mathcal{M}} \nabla R_t(\mathbf{x}, \mathbf{y}) a_{t,h}(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} - \sum_{\mathbf{p}_j \in P} \nabla R_t(\mathbf{x}, \mathbf{p}_j) a_{t,h}(\mathbf{p}_j) V_j \right|^2 \mathrm{d}\mu_{\mathbf{x}} \tag{E.4}$$

$$= \int_{\mathcal{M}} \left(\sum_{i=1}^n C_t u_i V_i B \right)^2 \mathrm{d}\mu_{\mathbf{x}}$$

$$\leq \frac{Ch^2}{t^2} \int_{\mathcal{M}} \left(\sum_{i=1}^n C_t |u_i| V_i R \left(\frac{|\mathbf{x} - \mathbf{p}_i|^2}{32t} \right) \right)^2 \mathrm{d}\mu_{\mathbf{x}}$$

$$\leq \frac{Ch^2}{t^2} \left(\sum_{i=1}^n u_i^2 V_i \right).$$

Combining Equation (E.2), (E.3) and (E.4), we have

$$\|\nabla(L_t a_{t,h} - L_{t,h} a_{t,h})\|_{L^2(\mathcal{M})}$$

$$= \left(\int_M |(L_t(a_{t,h}) - L_{t,h}(a_{t,h}))(\mathbf{x})|^2 \mathrm{d}\mu_{\mathbf{x}} \right)^{1/2}$$

$$\leq \frac{1}{t} \left(\int_{\mathcal{M}} (\nabla a_{t,h}(\mathbf{x}))^2 \left| \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} - \sum_{\mathbf{p}_j \in P} R_t(\mathbf{x}, \mathbf{p}_j) V_j \right|^2 \mathrm{d}\mu_{\mathbf{x}} \right)^{1/2}$$

$$\frac{1}{t} \left(\int_{\mathcal{M}} (a_{t,h}(\mathbf{x}))^2 \left| \int_{\mathcal{M}} \nabla_{\mathbf{x}} R_t(\mathbf{x}, \mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} - \sum_{\mathbf{p}_j \in P} \nabla_{\mathbf{x}} R_t(\mathbf{x}, \mathbf{p}_j) V_j \right|^2 \mathrm{d}\mu_{\mathbf{x}} \right)^{1/2}$$

$$+ \frac{1}{t} \left(\int_{\mathcal{M}} \left| \int_{\mathcal{M}} \nabla_{\mathbf{x}} R_t(\mathbf{x}, \mathbf{y}) a_{t,h}(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} - \sum_{\mathbf{p}_j \in P} \nabla_{\mathbf{x}} R_t(\mathbf{x}, \mathbf{p}_j) a_{t,h}(\mathbf{p}_j) V_j \right|^2 \mathrm{d}\mu_{\mathbf{x}} \right)^{1/2}$$

$$\leq \frac{Ch}{t^2} \left(\sum_{i=1}^n u_i^2 V_i \right)^{1/2} \leq \frac{Ch}{t^2} \|f\|_\infty$$

Using a similar argument, we obtain

$$\|\nabla(L_t c_{t,h} - L_{t,h} c_{t,h})\|_{L^2(\mathcal{M})} \leq \frac{Ch}{t^{3/2}} \|f\|_\infty,$$

and thus

$$\|\nabla(L_t u_{t,h} - L_{t,h} u_{t,h})\|_{L^2(\mathcal{M})} \leq \frac{Ch}{t^2} \|f\|_\infty. \tag{E.5}$$

Then the estimation is completed by putting (E.1) and (E.5) together.

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