NONLOCAL APPROXIMATION OF ELLIPTIC OPERATORS WITH ANISOTROPIC COEFFICIENTS ON MANIFOLD *

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Abstract. In this paper, we give an integral approximation for the elliptic operators with anisotropic coefficients on smooth manifold. Using the integral approximation, the elliptic equation is tranformed to an integral equation. The integral approximation preserves the symmetry and coercivity of the original elliptic operator. Based on these good properties, we prove the convergence between the solutions of the integral equation and the original elliptic equation.

1. Introduction. Recently, manifold model attracts more and more attentions in many applications, include data analysis and image processing [28, 27, 3, 7, 18, 15, 29, 13, 17, 6, 25, 33]. In the manifold model, data or images are assumed to be distributed in a low dimensional manifold embedded in a high dimensional Euclidean space. Differential operators on the manifold, particularly the elliptic operators, encode lots of intrinsic information of the manifold.

Besides the data analysis and image processing, PDEs on manifolds also arise in many different applications, including material science [5, 10], fluid flow [12, 14], biology and biophysics [2, 11, 26, 1]. Many methods have been developed to solve PDEs on curved surfaces embedded in $\mathbb{R}^3$, such as surface finite element method [9], level set method [4, 34], grid based particle method [20, 19] and closest point method [30, 24]. On the other hand, these methods do not apply in high dimensional problem directly.

In the past few years, many numerical methods to solve PDEs on manifold embedded in high dimensional space were developed. Liang et al. proposed to discretize the differential operators on point cloud by local least square approximations of the manifold [23]. Later, Lai et al. proposed local mesh method to approximate the differential operators on point cloud [16]. The main idea is to construct mesh locally around each point by using K nearest neighbors instead of constructing the global mesh. The other approach is so called point integral method [22, 21, 31, 32]. In the point integral method, the differential operators are approximated by integral operators. Then it is easy to discretize the integral operators in manifold since there is not any differential operators inside. The convergence of the point integral method for elliptic operators with isotropic coefficients has been proved [21].

In this paper, we consider to solve general elliptic operators with anisotropic coefficients on manifold $\mathcal{M}$. We assume that $\mathcal{M} \in C^2$ is a compact m-dimensional manifold isometrically embedded in $\mathbb{R}^d$ with the standard Euclidean metric and $m \leq d$. If $\mathcal{M}$ has boundary, the boundary, $\partial\mathcal{M}$ is also a C^2 smooth manifold.

Let $X : V \subset \mathbb{R}^m \to \mathcal{M} \subset \mathbb{R}^d$ be a local parametrization of $\mathcal{M}$ and $\theta \in V$. For any differentiable function $f : \mathcal{M} \to \mathbb{R}$, let $F(\theta) = f(X(\theta))$, define

$$D_k f(X(\theta)) = \sum_{i,j=1}^{m} g^{ij}(\theta)\frac{\partial X_k}{\partial \theta_i}(\theta)\frac{\partial F}{\partial \theta_j}(\theta), \quad k = 1, \cdots, d. \tag{1.1}$$

where $(g^{ij})_{i,j=1,\cdots,m} = G^{-1}$ and $G(\theta) = (g_{ij})_{i,j=1,\cdots,m}$ is the first fundamental form

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which is defined by

$$g_{ij}(\theta) = \sum_{k=1}^{d} \frac{\partial X_k}{\partial \theta_i}(\theta)\frac{\partial X_k}{\partial \theta_j}(\theta), \quad i,j = 1,\cdots,m. \tag{1.2}$$

The general second order elliptic PDE on manifold $\mathcal{M}$ has following form,

$$-\sum_{i,j=1}^{d} D_i(a_{ij}(\mathbf{x})D_j u(\mathbf{x})) = f(\mathbf{x}), \quad \mathbf{x} \in \mathcal{M} \tag{1.3}$$

The coeffcients $a_{ij}(\mathbf{x})$ and source term $f(\mathbf{x})$ are smooth functions of spatial variables, i.e.

$$a_{ij}, f \in C^1(\mathcal{M}), \quad i,j = 1,\cdots,d$$

The matrix $(a_{ij})_{i,j=1,\cdots,d}$ is symmetric and maps the tangent space $\mathcal{T}_{\mathbf{x}}$ into iteself and sastifies following elliptic condtion: there exist generic constants $0 < a_0, a_1 < \infty$ independent on $\mathbf{x}$ such that for any $\xi = [\xi_1, \cdots, \xi_d]^t \in \mathcal{T}_{\mathbf{x}}$,

$$a_0 \sum_{i=1}^{d} \xi_i^2 \le \sum_{i,j=1}^{d} a^{ij}(\mathbf{x})\xi_i\xi_j \le a_1 \sum_{i=1}^{d} \xi_i^2 \tag{1.4}$$

For any $\mathbf{x} \in \mathcal{M}$, the matrix $(a_{ij}(\mathbf{x}))$ gives a linear transform from $\mathbb{R}^d$ to $\mathbb{R}^d$, denoted as $A(\mathbf{x})$. The tangent space at $\mathbf{x}$, $\mathcal{T}_{\mathbf{x}}$, is a invariant subspace of $A(\mathbf{x})$. Confined on $\mathcal{T}_{\mathbf{x}}$, $A(\mathbf{x})$ introduces a linear transform from $\mathcal{T}_{\mathbf{x}}$ to $\mathcal{T}_{\mathbf{x}}$, denoted as $A_{\mathcal{T}}(\mathbf{x})$.

In [22, 21], the point integral method (PIM) was proposed for elliptic equation with isotropic coefficients, i.e.,

$$a_{ij}(\mathbf{x}) = p^2(\mathbf{x})\delta_{ij}, \tag{1.5}$$

where $p(\mathbf{x}) \ge C_0 > 0$ and

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

The main ingredient of the point integral method is to approximate the elliptic equation by an integral equation:

$$\frac{1}{t}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))p(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} - 2\int_{\partial\mathcal{M}} \frac{\partial u}{\partial \mathbf{n}}(\mathbf{y})\bar{R}_t(\mathbf{x},\mathbf{y})p(\mathbf{y})\mathrm{d}\tau_{\mathbf{y}} = \int_{\mathcal{M}} f(\mathbf{y})\frac{\bar{R}_t(\mathbf{x},\mathbf{y})}{p(\mathbf{y})}d\mu_{\mathbf{y}}, \tag{1.6}$$

where $\mathbf{n}$ is the out normal of $\partial\mathcal{M}$, $R_t(\mathbf{x},\mathbf{y})$ and $\bar{R}_t(\mathbf{x},\mathbf{y})$ are kernel functions given as following

$$R_t(\mathbf{x},\mathbf{y}) = C_t R\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right), \quad \bar{R}_t(\mathbf{x},\mathbf{y}) = C_t \bar{R}\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right) \tag{1.7}$$

where $C_t = \frac{1}{(4\pi t)^{k/2}}$ is the normalizing factor with $k = \dim(\mathcal{M})$. $R \in C^2(\mathbb{R}^+)$ be a positive function which is integrable over $[0, +\infty)$ and $\bar{R}(r) = \int_r^{+\infty} R(s)\mathrm{d}s$. The main advantage of the integral equation is that there is no differential operators in the equation. It is easy to be discretized from point clouds using numerical integration.

The main contribution of this paper is to generalize the point integral method to solve the general elliptic equation (1.3). The observation is to change the kernel function to

$$\bar{K}_t(\mathbf{x},\mathbf{y}) = \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) + \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y}) \tag{1.8}$$

$$K_t(\mathbf{x},\mathbf{y}) = \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) + \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y}) \tag{1.9}$$

where $|A_{\mathcal{T}}(\mathbf{x})|$ is the determinant of $A_{\mathcal{T}}(\mathbf{x})$ and

$$R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) = R\left(\frac{(x_m-y_m)a^{mn}(\mathbf{x})(x_n-y_n)}{4t}\right), R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y}) = R\left(\frac{(x_m-y_m)a^{mn}(\mathbf{y})(x_n-y_n)}{4t}\right)$$
$$\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) = \bar{R}\left(\frac{(x_m-y_m)a^{mn}(\mathbf{x})(x_n-y_n)}{4t}\right), \bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y}) = \bar{R}\left(\frac{(x_m-y_m)a^{mn}(\mathbf{y})(x_n-y_n)}{4t}\right)$$

with matrix $(a^{ij}(\mathbf{x}))_{i,j=1,\cdots,d}$ is the inverse of the coefficient matrix $(a_{ij}(\mathbf{x}))_{i,j=1,\cdots,d}$

Using above kernel function, we get an integral equation approximate the original elliptic equation (1.3),

$$\frac{1}{t}\int_{\mathcal{M}} K_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}} - \int_{\partial\mathcal{M}}\sum_{i,j=1}^{d} n_i(\mathbf{y})a_{ij}(\mathbf{y})D_j u(\mathbf{y})\bar{K}_t(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}} = \int_{\mathcal{M}}\bar{K}_t(\mathbf{x},\mathbf{y})f(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}, \tag{1.10}$$

Furthermore, under some mild assumption in Assumption 1.1, we prove that the solution of the integral equation (1.10) converges to the solution of the elliptic equation (1.3) as t goes to 0.

Assumption 1.1.

- Smoothness of the manifold: $\mathcal{M}, \partial\mathcal{M}$ are both compact and C^∞ smooth k-dimensional submanifolds isometrically embedded in a Euclidean space $\mathbb{R}^d$.
- Assumptions on the kernel function $R(r)$:
  - (a) Smoothness: $R \in C^2(\mathbb{R}^+)$;
  - (b) Nonnegativity: $R(r) \geq 0$ for any $r \geq 0$.
  - (c) Compact support: $R(r) = 0$ for $\forall r > 1$;
  - (d) Nondegeneracy: $\exists \delta_0 > 0$ so that $R(r) \geq \delta_0$ for $0 \leq r \leq \frac{1}{2}$.

Remark 1.1. *The assumption on the kernel function is very mild. The compact support assumption can be relaxed to exponentially decay, like Gaussian kernel. In the nondegeneracy assumption, 1/2 may be replaced by a positive number θ_0 with $0 < \theta_0 < 1$. Similar assumptions on the kernel function is also used in analysis the nonlocal diffusion problem [8].*

Under above assumptions, we prove the convergence which is stated in Theorem 1.1.

Theorem 1.1. *Let u be the solution to Problem (1.3) with $f \in C^1(\mathcal{M})$ and the vector u_t be the solution to the problem (1.10). Then there exists constants C and T_0 only depend on $\mathcal{M}$, such that for any $t \leq T_0$*

$$\|u-u_t\|_{H^1(\mathcal{M})} \leq Ct^{1/2}\|f\|_{C^1(\mathcal{M})}.$$

Remark 1.2. *Using the techniques in [31, 21], we can get the similar result for* $f \in H^1(\mathcal{M})$, *i.e.*

$$\|u - u_t\|_{H^1(\mathcal{M})} \le Ct^{1/2}\|f\|_{H^1(\mathcal{M})}.$$

To make the paper clear and concise, we only present the analysis for $f \in C^1(\mathcal{M})$. *The generalization for* $f \in H^1(\mathcal{M})$ *is straightforward following the similar arguments in [31, 21].*

The rest of this paper is organized as follows. In Section 2, we prove Theorem 1.1 based on the local truncation error estimate and the stability analysis. The local truncation error analysis is given in Section 3. The stability analysis is defered to Appendix, since it is similar to the result in our previous paper [31]. Finally, the conclusions and discussion of future work are provided in Section 4.

2. Proof of the Main Theorem (Theorem 1.1). To prove Theorem 1.1, we follow the standard argument in numerical analysis. First, we derive the truncation error of the integral approximation (1.10) in Theorem 2.1. Then, we use the stability estimate given in Theorem 2.2 and 2.3 to get the error estimate of the solution. In the truncation error, we have two terms, interior term and the boundary term. Corresponding to these two terms, we give two stability estimate respectively in Theorem 2.2 and 2.3.

Theorem 2.1. *Under the assumptions in Assumption 1.1, let* $u(\mathbf{x})$ *be the solution of the problem* (1.3) *and* $u_t(\mathbf{x})$ *be the solution of the corresponding integral equation* (1.10). *If* $u \in C^3(\mathcal{M})$, *then there exists constants* C, T_0 *depending only on* $\mathcal{M}$, $\partial\mathcal{M}$, *so that for any* $t \le T_0$,

$$\|L_t(u - u_t) - I_{bd}\|_{L^2(\mathcal{M})} \le Ct^{1/2}\|u\|_{C^3(\mathcal{M})}, \tag{2.1}$$

$$\|D\left(L_t(u - u_t) - I_{bd}\right)\|_{L^2(\mathcal{M})} \le C\|u\|_{C^3(\mathcal{M})}, \tag{2.2}$$

where

$$L_t u(\mathbf{x}) = \frac{1}{t}\int_{\mathcal{M}} K_t(\mathbf{x}, \mathbf{y})(u(\mathbf{x}) - u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}}. \tag{2.3}$$

and

$$\begin{aligned} I_{bd} = & \int_{\partial\mathcal{M}} n_i(\mathbf{y})a_{im}(\mathbf{x})\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}\bar{R}_t^{\mathbf{x}}(\mathbf{x}, \mathbf{y})\partial_{mn}u(\mathbf{y})(x_n - y_n)\mathrm{d}\tau_{\mathbf{y}} \\ & -2\int_{\partial\mathcal{M}} n_i(\mathbf{y})(y_k - x_k)\partial_k a_{ij}(\mathbf{x})\partial_j u(\mathbf{y})\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}\bar{R}_t^{\mathbf{x}}(\mathbf{x}, \mathbf{y})\mathrm{d}\tau_{\mathbf{y}} \\ & +\int_{\partial\mathcal{M}} n_i(\mathbf{y})a_{im}(\mathbf{x})\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}\bar{R}_t^{\mathbf{x}}(\mathbf{x}, \mathbf{y})\partial_{mn}u(\mathbf{y})(x_n - y_n)\mathrm{d}\tau_{\mathbf{y}} \\ & +\int_{\partial\mathcal{M}} n_k(\mathbf{y})a_{ij}(\mathbf{y})\partial_j u(\mathbf{y})\frac{\partial_i a^{mn}(\mathbf{y})(x_n - y_n)}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}a_{km}(\mathbf{x})\bar{R}_t^{\mathbf{x}}(\mathbf{x}, \mathbf{y})\mathrm{d}\tau_{\mathbf{y}}. \end{aligned} \tag{2.4}$$

The proof of this theorem will be given in Section 3

Next, we list two theorems about the stability.

Theorem 2.2. *Assume both the submanifolds* $\mathcal{M}$ *and* $\partial\mathcal{M}$ *are* C^∞, *and* $u(\mathbf{x})$ *solves the following equation*

$$-L_t u = r(\mathbf{x}) - \bar{r}$$

where $r \in H^1(\mathcal{M})$ *and* $\bar{r} = \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} r(\mathbf{x})\mathrm{d}\mu_{\mathbf{x}}$. *Then, there exist constants* $C > 0, T_0 > 0$ *independent on* t*, such that*

$$\|u\|_{H^1(\mathcal{M})} \leq C\left(\|r\|_{L^2(\mathcal{M})} + t\|\nabla r\|_{L^2(\mathcal{M})}\right)$$

as long as $t \leq T_0$.

For the boundary term I_{bd} in (2.4), the stablity result is given as follows.

THEOREM 2.3. *Assume both the submanifolds* $\mathcal{M}$ *and* $\partial\mathcal{M}$ *are* C^∞ *smooth. Let*

$$r(\mathbf{x}) = \sum_{i=1}^{d}\int_{\partial\mathcal{M}} b^i(\mathbf{y})(x_i - y_i)\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}$$

where $b^i(\mathbf{y}) \in L^\infty(\partial\mathcal{M})$ *for any* $1 \leq i \leq d$. *Assume* $u(\mathbf{x})$ *solves the following equation*

$$-L_t u = r - \bar{r},$$

where $\bar{r} = \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} r(\mathbf{x})\mathrm{d}\mu_{\mathbf{x}}$. *Then, there exist constants* $C > 0, T_0 > 0$ *independent on* t*, such that*

$$\|u\|_{H^1(\mathcal{M})} \leq C\sqrt{t}\max_{1\leq i\leq d}\left(\|b^i\|_\infty\right).$$

as long as $t \leq T_0$. The similar stability results have been given in our previous paper [31]. Above two theorems can be proved following the similar line as those in [31]. The details of the proof can be found in Appendix A and B respectively.

3. Proof of Theorem 2.1. *Proof.* Using the Gauss formula, we have

$$\begin{aligned}(3.1)\quad &\int_{\mathcal{M}} D_i(a_{ij}(\mathbf{y})D_j u(\mathbf{y}))\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\ &= -\int_{\mathcal{M}} a_{ij}(\mathbf{y})D_j u(\mathbf{y})D_i\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} + \int_{\partial\mathcal{M}} n_i(\mathbf{y})a_{ij}(\mathbf{y})D_j u(\mathbf{y})\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}.\end{aligned}$$

Substituting above expansion in the first term of (3.1), we get

$$\begin{aligned}(3.2)\quad &-\int_{\mathcal{M}} a_{ij}(\mathbf{y})D_j u(\mathbf{y})D_i\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\ &= -\frac{1}{2t}\int_{\mathcal{M}} a_{ij}(\mathbf{y})(\partial_{l'}\Phi^j g^{l'k'}\partial_j u(\mathbf{y}))\partial_{i'}\Phi^i g^{i'j'}\partial_{j'}\Phi^n a^{mn}(\mathbf{x})(x_m - y_m)R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}.\end{aligned}$$

The coefficients $a_{ij}(\mathbf{y})$ maps the tangent space $\mathcal{T}_{\mathbf{y}}$ into itself which means that there exists $c_{l'l}(\mathbf{y})$ such that

$$a_{ij}(\mathbf{y})\partial_{l'}\Phi^j = c_{l'l}(\mathbf{y})\partial_l\Phi^i.$$

Then

$$
\begin{aligned}
(3.3)\qquad & -\int_{\mathcal{M}} a_{ij}(\mathbf{y}) D_j u(\mathbf{y}) D_i \bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} \\
&= -\frac{1}{2t}\int_{\mathcal{M}} c_{l'l}(\mathbf{y}) \partial_l \Phi^i \partial_{i'} \Phi^i g^{l'k'} g^{i'j'} \partial_{j'} \Phi^n a^{mn}(\mathbf{x})(x_m - y_m) R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) \partial_{k'} u(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} \\
&= -\frac{1}{2t}\int_{\mathcal{M}} c_{l'j'}(\mathbf{y}) \partial_{j'} \Phi^n g^{l'k'} a^{mn}(\mathbf{x})(x_m - y_m) R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) \partial_{k'} u(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} \\
&= -\frac{1}{2t}\int_{\mathcal{M}} a_{nl}(\mathbf{y}) \partial_{l'} \Phi^l g^{l'k'} a^{mn}(\mathbf{x})(x_m - y_m) R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) \partial_{k'} u(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} \\
&= -\frac{1}{2t}\int_{\mathcal{M}} a_{nl}(\mathbf{y}) a^{mn}(\mathbf{x})(x_m - y_m) R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) D_l u(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} \\
&= -\frac{1}{2t}\int_{\mathcal{M}} (x_l - y_l) D_l u(\mathbf{y}) R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} \\
&\quad -\frac{1}{2t}\int_{\mathcal{M}} (a_{nl}(\mathbf{y}) - a_{nl}(\mathbf{x})) a^{mn}(\mathbf{x})(x_m - y_m) R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) D_l u(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}}
\end{aligned}
$$

Notice that

$$
\begin{aligned}
D_n \bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) =& \frac{1}{2t} \partial_{i'} \Phi^n g^{i'j'} \partial_{j'} \Phi^l a^{ml}(\mathbf{x})(x_m - y_m) R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) \\
(3.4)\qquad =& \frac{1}{2t} \partial_{i'} \Phi^n g^{i'j'} \partial_{j'} \Phi^l a^{ml}(\mathbf{y}) \partial_{m'} \Phi^m (\alpha_{m'} - \beta_{m'}) R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) + O(1)
\end{aligned}
$$

Since $a^{ml}(\mathbf{y})$ also maps the tangent space $T_{\mathbf{y}}\mathcal{M}$ into itself, there exists $d_{l'l}(\mathbf{y})$ such that

$$
a^{ml}(\mathbf{y}) \partial_{m'} \Phi^m = d_{m'l'}(\mathbf{y}) \partial_{l'} \Phi^l .
$$

It follows that

$$
\begin{aligned}
(3.5)\qquad D_n \bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) =& \frac{1}{2t} d_{m'l'}(\mathbf{y}) \partial_{l'} \Phi^l \partial_{j'} \Phi^l g^{i'j'} \partial_{i'} \Phi^n (\alpha_{m'} - \beta_{m'}) R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) + O(1) \\
=& \frac{1}{2t} d_{m'i'}(\mathbf{y}) \partial_{i'} \Phi^n (\alpha_{m'} - \beta_{m'}) R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) + O(1) \\
=& \frac{1}{2t} a^{mn}(\mathbf{y}) \partial_{m'} \Phi^m (\alpha_{m'} - \beta_{m'}) R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) + O(1) \\
=& \frac{1}{2t} a^{mn}(\mathbf{y}) (x_m - y_m) R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) + O(1) \\
=& \frac{1}{2t} a^{mn}(\mathbf{x}) (x_m - y_m) R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) + O(1).
\end{aligned}
$$

The last term of (3.3) becomes

$$
\begin{aligned}
(3.6)\qquad & \frac{1}{2t}\int_{\mathcal{M}} (a_{nl}(\mathbf{y}) - a_{nl}(\mathbf{x})) a^{mn}(\mathbf{x})(x_m - y_m) R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) D_l u(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} \\
&= \int_{\mathcal{M}} (a_{nl}(\mathbf{y}) - a_{nl}(\mathbf{x})) D_n^{\mathbf{y}} \bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) D_l u(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} + O(\sqrt{t}) \\
&= -\int_{\mathcal{M}} D_n a_{nl}(\mathbf{y}) D_l u(\mathbf{y}) \bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}} \\
&\quad + \int_{\partial\mathcal{M}} n_n(\mathbf{y}) (a_{nl}(\mathbf{y}) - a_{nl}(\mathbf{x})) \bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) D_l u(\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}} + O(\sqrt{t}).
\end{aligned}
$$

Now, we turn to estimate the first term of (3.3). In this step, we need the help of Taylor's expansion of $u(\mathbf{x})$ at $\mathbf{y}$,

$$u(\mathbf{x})-u(\mathbf{y}) = (x_j-y_j)D_ju(\mathbf{y})+\frac{1}{2}D_mD_nu(\mathbf{y})(x_m-y_m)(x_n-y_n)+O(\|\mathbf{x}-\mathbf{y}\|^3) \tag{3.7}$$

This expansion gives immediately

$$\begin{aligned} &-\frac{1}{2t}\int_{\mathcal{M}}(x_j-y_j)D_ju(\mathbf{y})R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\ =&-\frac{1}{2t}\int_{\mathcal{M}}R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}} \\ &+\frac{1}{4t}\int_{\mathcal{M}}R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_mD_nu(\mathbf{y})(x_m-y_m)(x_n-y_n)\mathrm{d}\mu_{\mathbf{y}}+O(\sqrt{t}). \end{aligned} \tag{3.8}$$

Next, we focus on the second term of (3.8). It follows from (3.5) that

$$\begin{aligned} a_{im}(\mathbf{x})D_i\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) =&\frac{1}{2t}a_{mi}(\mathbf{x})a^{m'i}(\mathbf{x})(x_{m'}-y_{m'})R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})+O(1) \\ =&\frac{1}{2t}a_{mi}(\mathbf{x})a^{m'i}(\mathbf{x})(x_{m'}-y_{m'})R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})+O(1) \\ =&\frac{1}{2t}(x_m-y_m)R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})+O(1). \end{aligned} \tag{3.9}$$

The second term of (3.8) is calculated as

$$\begin{aligned} &\frac{1}{4t}\int_{\mathcal{M}}R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_mD_nu(\mathbf{y})(x_m-y_m)(x_n-y_n)\mathrm{d}\mu_{\mathbf{y}} \\ =&\frac{1}{2}\int_{\mathcal{M}}a_{im}(\mathbf{x})D_i\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_mD_nu(\mathbf{y})(x_n-y_n)\mathrm{d}\mu_{\mathbf{y}} \\ =&\frac{1}{2}\int_{\mathcal{M}}a_{im}(\mathbf{x})(\partial_{i'}\Phi^i g^{i'j'}\partial_{j'}\Phi^n)D_mD_nu(\mathbf{y})\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\ &+\frac{1}{2}\int_{\partial\mathcal{M}}n_i(\mathbf{y})a_{im}(\mathbf{x})\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_mD_nu(\mathbf{y})(x_n-y_n)\mathrm{d}\mu_{\mathbf{y}}. \end{aligned} \tag{3.10}$$

Notice that

$$\begin{aligned} &a_{im}(\mathbf{x})(\partial_{i'}\Phi^i g^{i'j'}\partial_{j'}\Phi^n)D_m \\ =&a_{im}(\mathbf{y})(\partial_{i'}\Phi^i g^{i'j'}\partial_{j'}\Phi^n)(\partial_{i''}\Phi^m g^{i''j''}\partial_{j''})+O(\sqrt{t}) \\ =&c_{i''l}\partial_l\Phi^i\partial_{i'}\Phi^i g^{i'j'}\partial_{j'}\Phi^n g^{i''j''}\partial_{j''}+O(\sqrt{t}) \\ =&c_{i''l}\partial_l\Phi^n g^{i''j''}\partial_{j''}+O(\sqrt{t}) \\ =&a_{mn}\partial_{i''}\Phi^m g^{i''j''}\partial_{j''}+O(\sqrt{t}) \\ =&a_{mn}D_m+O(\sqrt{t}) \end{aligned}$$

From 3.10, we obtain

$$\begin{aligned} &\frac{1}{4t}\int_{\mathcal{M}}R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_mD_nu(\mathbf{y})(x_m-y_m)(x_n-y_n)\mathrm{d}\mu_{\mathbf{y}} \\ =&\frac{1}{2}\int_{\mathcal{M}}a_{mn}(\mathbf{x})D_mD_nu(\mathbf{y})\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\ &+\frac{1}{2}\int_{\partial\mathcal{M}}n_i(\mathbf{y})a_{im}(\mathbf{x})\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_mD_nu(\mathbf{y})(x_n-y_n)\mathrm{d}\mu_{\mathbf{y}}+O(\sqrt{t}). \end{aligned} \tag{3.11}$$

Now, using (3.1), (3.3), (3.8) and (3.11), we get

$$
\begin{aligned}
&\int_{\mathcal{M}} D_i(a_{ij}(\mathbf{y})D_j u(\mathbf{y}))\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\
&= -\frac{1}{2t}\int_{\mathcal{M}} R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}} + \frac{1}{2}\int_{\mathcal{M}} a_{ij}(\mathbf{x})D_m D_n u(\mathbf{y})\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\
&\quad + \int_{\mathcal{M}} D_i a_{ij}(\mathbf{x})D_j u(\mathbf{y})\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} + \int_{\partial\mathcal{M}} n_i(\mathbf{y})a_{ij}(\mathbf{y})D_j u(\mathbf{y})\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\
&\quad + B.T.1 + O(\sqrt{t})
\end{aligned}
\tag{3.12}
$$

where

$$
\begin{aligned}
B.T.1 = &\frac{1}{2}\int_{\partial\mathcal{M}} n_i(\mathbf{y})a_{im}(\mathbf{x})\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_m D_n u(\mathbf{y})(x_n-y_n)\mathrm{d}\tau_{\mathbf{y}} \\
&-\int_{\partial\mathcal{M}} n_i(\mathbf{y})(a_{ij}(\mathbf{y})-a_{ij}(\mathbf{x}))\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_j u(\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}
\end{aligned}
\tag{3.13}
$$

Now, we change the kernel function to $\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})$ and get

$$
\begin{aligned}
&\int_{\mathcal{M}} D_i(a_{ij}(\mathbf{y})D_j u(\mathbf{y}))\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\
&= -\int_{\mathcal{M}} a_{ij}(\mathbf{y})D_j u(\mathbf{y})D_i\left(\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\right)\mathrm{d}\mu_{\mathbf{y}} \\
&\quad + \int_{\partial\mathcal{M}} n_i(\mathbf{y})a_{ij}(\mathbf{y})D_j u(\mathbf{y})\frac{\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\mathrm{d}\tau_{\mathbf{y}}.
\end{aligned}
\tag{3.14}
$$

Direct calculation gives that the first term of (3.14) becomes

$$
\begin{aligned}
&-\int_{\mathcal{M}} a_{ij}(\mathbf{y})D_j u(\mathbf{y})D_i\left(\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\right)\mathrm{d}\mu_{\mathbf{y}} \\
&= -\frac{1}{2t}\int_{\mathcal{M}} \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}a_{ij}(\mathbf{y})D_j u(\mathbf{y})\partial_{i'}\Phi^i g^{i'j'}\partial_{j'}\Phi^n a^{mn}(\mathbf{y})(x_m-y_m)R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\
&\quad + \frac{1}{4t}\int_{\mathcal{M}} \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}a_{ij}(\mathbf{y})\partial_j u(\mathbf{y})D_i a^{mn}(\mathbf{y})(x_m-y_m)(x_n-y_n)R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\
&\quad - \int_{\mathcal{M}} a_{ij}(\mathbf{y})\partial_j u(\mathbf{y})\partial_i\left(\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\right)\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}.
\end{aligned}
\tag{3.15}
$$

Next, we will estimate the three terms in (3.15) one by one.

$$
\begin{aligned}
&-\frac{1}{2t}\int_{\mathcal{M}} \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}a_{ij}(\mathbf{y})D_j u(\mathbf{y})\partial_{i'}\Phi^i g^{i'j'}\partial_{j'}\Phi^n a^{mn}(\mathbf{y})(x_m-y_m)R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\
&= -\frac{1}{2t}\int_{\mathcal{M}} (x_j-y_j)D_j u(\mathbf{y})\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\
&= -\frac{1}{2t}\int_{\mathcal{M}} \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}} \\
&\quad + \frac{1}{4t}\int_{\mathcal{M}} \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})D_m D_n u(\mathbf{y})(x_m-y_m)(x_n-y_n)\mathrm{d}\mu_{\mathbf{y}} + O(\sqrt{t}).
\end{aligned}
\tag{3.16}
$$

The first equality is from (3.5) In the second equality, we use the Taylor's expansion (3.7). We keep the first term of (3.16) and the second term can be further calculated as

$$
\begin{aligned}
(3.17)\quad & \frac{1}{4t}\int_{\mathcal{M}} \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}} R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y}) D_m D_n u(\mathbf{y})(x_m-y_m)(x_n-y_n)\mathrm{d}\mu_{\mathbf{y}} \\
&= \frac{1}{4t}\int_{\mathcal{M}} \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}} R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) D_m D_n u(\mathbf{y})(x_m-y_m)(x_n-y_n)\mathrm{d}\mu_{\mathbf{y}} + O(\sqrt{t}) \\
&= \frac{1}{2}\int_{\mathcal{M}} a_{ij}(\mathbf{x}) D_i D_j u(\mathbf{y}) \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}} \bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\
&\quad + \frac{1}{2}\int_{\partial\mathcal{M}} n_i(\mathbf{y}) a_{im}(\mathbf{x}) \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}} \bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) D_m D_n u(\mathbf{y})(x_n-y_n)\mathrm{d}\mu_{\mathbf{y}} + O(\sqrt{t}) \\
&= \frac{1}{2}\int_{\mathcal{M}} a_{ij}(\mathbf{x}) D_i D_j u(\mathbf{y}) \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}} \bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\
&\quad + \frac{1}{2}\int_{\partial\mathcal{M}} n_i(\mathbf{y}) a_{im}(\mathbf{x}) \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}} \bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) D_m D_n u(\mathbf{y})(x_n-y_n)\mathrm{d}\mu_{\mathbf{y}} + O(\sqrt{t}).
\end{aligned}
$$

To get the second equality, we use the same calculation as that in (3.11).

The second term of (3.15) is calculated as

$$
\begin{aligned}
(3.18)\quad & \frac{1}{4t}\int_{\mathcal{M}} a_{ij}(\mathbf{y}) D_j u(\mathbf{y}) \frac{D_i a^{mn}(\mathbf{y})(x_m-y_m)(x_n-y_n)}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}} R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\
&= \frac{1}{4t}\int_{\mathcal{M}} a_{ij}(\mathbf{y}) D_j u(\mathbf{y}) \frac{D_i a^{mn}(\mathbf{y})(x_m-y_m)(x_n-y_n)}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}} R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} + O(\sqrt{t}) \\
&= \frac{1}{2}\int_{\mathcal{M}} a_{ij}(\mathbf{y}) D_j u(\mathbf{y}) \frac{D_i a^{mn}(\mathbf{y})(x_n-y_n)}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}} a_{km}(\mathbf{x}) D_k \bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} + O(\sqrt{t}) \\
&= \frac{1}{2}\int_{\mathcal{M}} \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}} a_{ij}(\mathbf{y}) D_j u(\mathbf{y}) D_i a^{mn}(\mathbf{y}) a_{km}(\mathbf{y}) (\partial_{i'}\Phi^k g^{i'j'}\partial_{j'}\Phi^n) \bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\
&\quad + \frac{1}{2}\int_{\partial\mathcal{M}} n_k(\mathbf{y}) a_{ij}(\mathbf{y}) D_j u(\mathbf{y}) \frac{\partial_i a^{mn}(\mathbf{y})(x_n-y_n)}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}} a_{km}(\mathbf{x}) \bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} + O(\sqrt{t}).
\end{aligned}
$$

In addition, we have that

$$
(3.19)\qquad D_i a^{mn}(\mathbf{y}) a_{km}(\mathbf{y}) (\partial_{i'}\Phi^k g^{i'j'}\partial_{j'}\Phi^n) = \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}} D_i \sqrt{|A_{\mathcal{T}}(\mathbf{x})|}.
$$

The derivation of this equation can be found in Appendix C.

Using above equation, we obtain

$$
\begin{aligned}
(3.20)\quad & \frac{1}{2}\int_{\mathcal{M}} \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}} a_{ij}(\mathbf{y}) D_j u(\mathbf{y}) D_i a^{mn}(\mathbf{y}) a_{km}(\mathbf{y}) (\partial_{i'}\Phi^k g^{i'j'}\partial_{j'}\Phi^n) \bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\
&= -\int_{\mathcal{M}} a_{ij}(\mathbf{y}) D_j u(\mathbf{y}) \frac{D_i \sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}{|A_{\mathcal{T}}(\mathbf{y})|} \bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} + O(\sqrt{t}) \\
&= \int_{\mathcal{M}} a_{ij}(\mathbf{y}) D_j u(\mathbf{y}) D_i \left(\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\right) \bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} + O(\sqrt{t}).
\end{aligned}
$$

Using (3.14), (3.15), (3.16), (3.17), (3.18) and (3.20),

$$
\begin{aligned}
(3.21)\qquad & \int_{\mathcal{M}} D_i(a_{ij}(\mathbf{y})D_j u(\mathbf{y}))\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\
&= -\frac{1}{2t}\int_{\mathcal{M}} \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}} R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}} \\
&\quad +\frac{1}{2}\int_{\mathcal{M}} a_{ij}(\mathbf{x})D_iD_j u(\mathbf{y})\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\
&\quad +\int_{\partial\mathcal{M}} n_i(\mathbf{y})a_{ij}(\mathbf{y})D_j u(\mathbf{y})\frac{\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\mathrm{d}\tau_{\mathbf{y}} + B.T.2 + O(\sqrt{t})
\end{aligned}
$$

where

$$
\begin{aligned}
B.T.2 &= \frac{1}{2}\int_{\partial\mathcal{M}} n_i(\mathbf{y})a_{im}(\mathbf{x})\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_mD_n u(\mathbf{y})(x_n-y_n)\mathrm{d}\tau_{\mathbf{y}} \\
(3.22)\qquad &\quad +\frac{1}{2}\int_{\partial\mathcal{M}} n_k(\mathbf{y})a_{ij}(\mathbf{y})D_j u(\mathbf{y})\frac{D_i a^{mn}(\mathbf{y})(x_n-y_n)}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}a_{km}(\mathbf{x})\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}
\end{aligned}
$$

Now, (3.12) and (3.21) imply that

$$
\begin{aligned}
(3.23)\quad & \int_{\mathcal{M}} D_i(a_{ij}(\mathbf{y})D_j u(\mathbf{y}))\left(\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) + \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\right)\mathrm{d}\mu_{\mathbf{y}} \\
&= -\frac{1}{2t}\int_{\mathcal{M}}\left(\frac{R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}} + \frac{R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\right)(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}} \\
&\quad +\frac{1}{2}\int_{\mathcal{M}} D_i(a_{ij}(\mathbf{x})D_j u(\mathbf{y}))\left(\frac{\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}} + \frac{\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\right)\mathrm{d}\mu_{\mathbf{y}} \\
&\quad +\int_{\partial\mathcal{M}} n_i(\mathbf{y})a_{ij}(\mathbf{y})D_j u(\mathbf{y})\left(\frac{\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}} + \frac{\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\right)\mathrm{d}\tau_{\mathbf{y}} + I_{bd} + O(\sqrt{t})
\end{aligned}
$$

where

$$
\begin{aligned}
(3.24)\quad I_{bd} &= \frac{1}{2}\int_{\partial\mathcal{M}} n_i(\mathbf{y})a_{im}(\mathbf{x})\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_mD_n u(\mathbf{y})(x_n-y_n)\mathrm{d}\tau_{\mathbf{y}} \\
&\quad -\int_{\partial\mathcal{M}} n_i(\mathbf{y})(a_{ij}(\mathbf{y})-a_{ij}(\mathbf{x}))\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_j u(\mathbf{y})\mathrm{d}\tau_{\mathbf{y}} \\
&\quad +\frac{1}{2}\int_{\partial\mathcal{M}} n_i(\mathbf{y})a_{im}(\mathbf{x})\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_mD_n u(\mathbf{y})(x_n-y_n)\mathrm{d}\tau_{\mathbf{y}} \\
&\quad +\frac{1}{2}\int_{\partial\mathcal{M}} n_k(\mathbf{y})a_{ij}(\mathbf{y})D_j u(\mathbf{y})\frac{D_i a^{mn}(\mathbf{y})(x_n-y_n)}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}a_{km}(\mathbf{x})\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}.
\end{aligned}
$$

Finally, it follows from (3.23) that

$$\begin{aligned}&\int_{\mathcal{M}} D_i(a_{ij}(\mathbf{y})D_j u(\mathbf{y}))\left(\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})+\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\right)\mathrm{d}\mu_{\mathbf{y}}\\ =&-\frac{1}{t}\int_{\mathcal{M}}\left(\frac{R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}+\frac{R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\right)(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}}\\ &+2\int_{\partial\mathcal{M}} n_i(\mathbf{y})a_{ij}(\mathbf{y})D_j u(\mathbf{y})\left(\frac{\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}+\frac{\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\right)\mathrm{d}\tau_{\mathbf{y}}+2B.T.+O(\sqrt{t}).\end{aligned}$$

□

4. Conclusion. In this paper, we give an integral approximation for the elliptic operators with anisotropic coefficients on smooth manifold. The integral approximation is proved to preserve the symmetry and coercivity of the original elliptic operator. Using the integral approximation, we get an integral equation which approximates the original elliptic equation. Moreover, we prove the convergence between the solutions of the integral equation and the original elliptic equation.

One advantage of the integral equation is that there is not any differential operators inside. Then it is easy to develop the numerical scheme in high dimensional point cloud.

Appendix A. Proof of Theorem 2.2.

In the proof we need two technical lemmas which have been proved in [31].

LEMMA A.1. *If t is small enough, then for any function $u \in L^2(\mathcal{M})$, there exists a constant $C > 0$ independent on t and u, such that*

$$\int_{\mathcal{M}}\int_{\mathcal{M}} R\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{32t}\right)(u(\mathbf{x})-u(\mathbf{y}))^2\mathrm{d}\mu_{\mathbf{x}}\mathrm{d}\mu_{\mathbf{y}} \le C\int_{\mathcal{M}}\int_{\mathcal{M}} R\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right)(u(\mathbf{x})-u(\mathbf{y}))^2\mathrm{d}\mu_{\mathbf{x}}\mathrm{d}\mu_{\mathbf{y}}.$$

LEMMA A.2. *Assume both $\mathcal{M}$ and $\partial\mathcal{M}$ are C^∞. There exists a constant $C > 0$ independent on t so that for any function $u \in L_2(\mathcal{M})$ with $\int_{\mathcal{M}} u(\mathbf{x})\mathrm{d}\mu_{\mathbf{x}} = 0$ and for any sufficient small t*

$$\frac{1}{t}\int_{\mathcal{M}}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))^2\mathrm{d}\mu_{\mathbf{x}}\mathrm{d}\mu_{\mathbf{y}} \ge C\|u\|^2_{L_2(\mathcal{M})} \tag{A.1}$$

One direct corollary of above two lemmas is that the conclusion in Lemma A.2 still holds if the kernel is replaced by $K(\mathbf{x},\mathbf{y},t)$.

THEOREM A.3. *Assume both $\mathcal{M}$ and $\partial\mathcal{M}$ are C^∞. There exists a constant $C > 0$ independent on t so that for any function $u \in L_2(\mathcal{M})$ with $\int_{\mathcal{M}} u = 0$ and for any sufficient small t*

$$\int_{\mathcal{M}}\int_{\mathcal{M}} K(\mathbf{x},\mathbf{y},t)(u(\mathbf{x})-u(\mathbf{y}))^2\mathrm{d}\mu_{\mathbf{x}}\mathrm{d}\mu_{\mathbf{y}} \ge C\|u\|^2_{L_2(\mathcal{M})} \tag{A.2}$$

To control the derivative, we also need following theorem.

THEOREM A.4. *For any function $u \in L^2(\mathcal{M})$, there exists a constant $C > 0$ independent on t and u, such that*

$$\int_{\mathcal{M}}\int_{\mathcal{M}} K(\mathbf{x},\mathbf{y},t)(u(\mathbf{x})-u(\mathbf{y}))^2 \mathrm{d}\mu_{\mathbf{x}}\mathrm{d}\mu_{\mathbf{y}} \geq C\int_{\mathcal{M}} |\nabla v|^2 \mathrm{d}\mu_{\mathbf{x}} \tag{A.3}$$

where

$$v(\mathbf{x}) = \frac{C_t}{w_t(\mathbf{x})}\int_{\mathcal{M}} K(\mathbf{x},\mathbf{y},t)u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}, \tag{A.4}$$

and $w_t(\mathbf{x}) = C_t\int_{\mathcal{M}} K(\mathbf{x},\mathbf{y},t)\mathrm{d}\mu_{\mathbf{y}}$.

Proof. We start with the evaluation of the x^i component of ∇v, $1 \leq i \leq d$.

$$\begin{aligned}
\nabla^i v(\mathbf{x}) &= \frac{1}{w_t^2(\mathbf{x})}\int_{\mathcal{M}}\int_{\mathcal{M}} K(\mathbf{x},\mathbf{y}',t)\nabla^i_{\mathbf{x}} K(\mathbf{x},\mathbf{y},t)u(\mathbf{y})\mathrm{d}\mu'_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{y}} \\
&\quad -\frac{1}{w_t^2(\mathbf{x})}\int_{\mathcal{M}}\int_{\mathcal{M}} K(\mathbf{x},\mathbf{y},t)\nabla^i_{\mathbf{x}} K(\mathbf{x},\mathbf{y}',t)u(\mathbf{y})\mathrm{d}\mu'_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{y}} \\
&= \frac{1}{w_t^2(\mathbf{x})}\int_{\mathcal{M}}\int_{\mathcal{M}} K(\mathbf{x},\mathbf{y}',t)\nabla^i_{\mathbf{x}} K(\mathbf{x},\mathbf{y},t)(u(\mathbf{y})-u(\mathbf{y}'))\mathrm{d}\mu'_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{y}} \\
&= \frac{1}{w_t^2(\mathbf{x})}\int_{\mathcal{M}}\int_{\mathcal{M}} \mathcal{Q}(\mathbf{x},\mathbf{y},\mathbf{y}',t)(u(\mathbf{y})-u(\mathbf{y}'))\mathrm{d}\mu'_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{y}}
\end{aligned}$$

where $\mathcal{Q}_i(\mathbf{x},\mathbf{y},\mathbf{y}',t) = K(\mathbf{x},\mathbf{y}',t)\nabla^i_{\mathbf{x}} K(\mathbf{x},\mathbf{y},t)$.

Notice that $\mathcal{Q}_i(\mathbf{x},\mathbf{y},\mathbf{y}',t) = 0$ when $|\mathbf{x}-\mathbf{y}|^2 \geq 4t/\lambda$ or $|\mathbf{x}-\mathbf{y}'|^2 \geq 4t/\lambda$. This implies that $\mathcal{Q}_i(\mathbf{x},\mathbf{y},\mathbf{y}',t) = 0$ when $|\mathbf{y}-\mathbf{y}'|^2 \geq 16t/\lambda$ or $|\mathbf{x}-\frac{\mathbf{y}+\mathbf{y}'}{2}|^2 \geq 4t/\lambda$. Thus from the assumption on R, we have

$$\mathcal{Q}_i(\mathbf{x},\mathbf{y},\mathbf{y}';t)^2 \leq \frac{1}{\delta_0^2}\mathcal{Q}_i(\mathbf{x},\mathbf{y},\mathbf{y}';t)^2 R\left(\frac{\lambda\|\mathbf{y}-\mathbf{y}'\|^2}{32t}\right) R\left(\frac{\lambda\|\mathbf{x}-\frac{\mathbf{y}+\mathbf{y}'}{2}\|^2}{8t}\right).$$

We can upper bound the norm of ∇v as follows:

$$\begin{aligned}
|\nabla v(\mathbf{x})|^2 &= \frac{1}{w_t^4(\mathbf{x})}\sum_{i=1}^d \left(\int_{\mathcal{M}}\int_{\mathcal{M}} \mathcal{Q}_i(\mathbf{x},\mathbf{y},\mathbf{y}';t)(u(\mathbf{y})-u(\mathbf{y}'))\mathrm{d}\mathbf{y}'\mathrm{d}\mathbf{y}\right)^2 \\
\leq &\frac{1}{w_t^4(\mathbf{x})}\sum_{i=1}^d \int_{\mathcal{M}}\int_{\mathcal{M}} \mathcal{Q}_i^2(\mathbf{x},\mathbf{y},\mathbf{y}';t)\left(R\left(\frac{\lambda\|\mathbf{y}-\mathbf{y}'\|^2}{32t}\right) R\left(\frac{\lambda\|\mathbf{x}-\frac{\mathbf{y}+\mathbf{y}'}{2}\|^2}{8t}\right)\right)^{-1}\mathrm{d}\mu'_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{y}} \\
&\int_{\mathcal{M}}\int_{\mathcal{M}} R\left(\frac{\lambda\|\mathbf{x}-\frac{\mathbf{y}+\mathbf{y}'}{2}\|^2}{8t}\right) R\left(\frac{\lambda\|\mathbf{y}-\mathbf{y}'\|^2}{32t}\right)(u(\mathbf{y})-u(\mathbf{y}'))^2\mathrm{d}\mu'_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{y}} \\
= &\frac{C}{t}\int_{\mathcal{M}}\int_{\mathcal{M}} t\sum_{i=1}^d \mathcal{Q}_i^2(\mathbf{x},\mathbf{y},\mathbf{y}';t)\mathrm{d}\mu'_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{y}} \\
&\int_{\mathcal{M}}\int_{\mathcal{M}} R\left(\frac{\lambda|\mathbf{x}-\frac{\mathbf{y}+\mathbf{y}'}{2}|^2}{8t}\right) R\left(\frac{\lambda|\mathbf{y}-\mathbf{y}'|^2}{32t}\right)(u(\mathbf{y})-u(\mathbf{y}'))^2\mathrm{d}\mu'_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{y}}.
\end{aligned}$$

By direct calculation, it is easy to check that

$$\int_{\mathcal{M}}\int_{\mathcal{M}} t\sum_{i=1}^d \mathcal{Q}_i^2(\mathbf{x},\mathbf{y},\mathbf{y}';t)\mathrm{d}\mu'_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{y}} \leq CC_t^2$$

where $C > 0$ is a generic constant.

Finally, we have

$$
\begin{aligned}
&\int_{\mathcal{M}} |\nabla v(\mathbf{x})|^2 \mathrm{d}\mu_{\mathbf{x}} \\
\leq& \frac{CC_t^2}{t} \int_{\mathcal{M}} \left(\int_{\mathcal{M}} \int_{\mathcal{M}} R\left(\frac{\lambda \|\mathbf{x} - \frac{\mathbf{y}+\mathbf{y}'}{2}\|^2}{8t} \right) R\left(\frac{\lambda \|\mathbf{y}-\mathbf{y}'\|^2}{32t} \right) (u(\mathbf{y}) - u(\mathbf{y}'))^2 \mathrm{d}\mu'_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{y}} \right) \mathrm{d}\mu_{\mathbf{x}} \\
=& \frac{CC_t^2}{t} \int_{\mathcal{M}} \int_{\mathcal{M}} \left(\int_{\mathcal{M}} R\left(\frac{\lambda \|\mathbf{x} - \frac{\mathbf{y}+\mathbf{y}'}{2}\|^2}{8t} \right) \mathrm{d}\mu_{\mathbf{x}} \right) R\left(\frac{\lambda \|\mathbf{y}-\mathbf{y}'\|^2}{32t} \right) (u(\mathbf{y}) - u(\mathbf{y}'))^2 \mathrm{d}\mu'_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{y}} \\
\leq& \frac{CC_t}{t} \int_{\mathcal{M}} \int_{\mathcal{M}} R\left(\frac{\lambda \|\mathbf{y}-\mathbf{y}'\|^2}{32t} \right) (u(\mathbf{y}) - u(\mathbf{y}'))^2 \mathrm{d}\mu'_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{y}}.
\end{aligned}
$$

Using Lemma A.1,

$$
\begin{aligned}
&\int_{\mathcal{M}} |\nabla v(\mathbf{x})|^2 \mathrm{d}\mu_{\mathbf{x}} \\
\leq& \frac{CC_t}{t} \int_{\mathcal{M}} \int_{\mathcal{M}} R\left(\frac{\Lambda \|\mathbf{y}-\mathbf{y}'\|^2}{2t} \right) (u(\mathbf{y}) - u(\mathbf{y}'))^2 \mathrm{d}\mu'_{\mathbf{y}} \mathrm{d}\mu_{\mathbf{y}} \\
\leq& C \int_{\mathcal{M}} \int_{\mathcal{M}} K(\mathbf{x}, \mathbf{y}, t) (u(\mathbf{x}) - u(\mathbf{y}))^2 \mathrm{d}\mu_{\mathbf{x}} \mathrm{d}\mu_{\mathbf{y}}.
\end{aligned}
$$

□

With Theorem A.4 and A.3, the proof of Theorem 2.2 is straightforward.

Proof. of Theorem 2.2

Using Theorem A.3, we have

$$
\begin{aligned}
\|u\|^2_{L^2(\mathcal{M})} \leq& C \left\langle u, L_t u \right\rangle = C \int_{\mathcal{M}} u(\mathbf{x})(r(\mathbf{x}) - \bar{r}) \mathrm{d}\mu_{\mathbf{x}} \\
\leq& C \|u\|_{L^2(\mathcal{M})} \|r\|_{L^2(\mathcal{M})}.
\end{aligned}
\tag{A.5}
$$

To show the last inequality, we use the fact that

$$
|\bar{r}| = \frac{1}{|\mathcal{M}|} \left| \int_{\mathcal{M}} r(\mathbf{x}) \mathrm{d}\mu_{\mathbf{x}} \right| \leq C \|r\|_{L^2(\mathcal{M})}.
$$

(A.5) implies that

$$
\|u\|_{L^2(\mathcal{M})} \leq C \|r\|_{L^2(\mathcal{M})}.
$$

Now we turn to estimate $\|\nabla u\|_{L^2(\mathcal{M})}$. Notice that we have the following expression for u, since u satisfies the integral equation (1.10).

$$
u(\mathbf{x}) = v(\mathbf{x}) + \frac{t}{w_t(\mathbf{x})} (r(\mathbf{x}) - \bar{r}),
$$

where

$$
v(\mathbf{x}) = \frac{1}{w_t(\mathbf{x})} \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) u(\mathbf{y}) \mathrm{d}\mu_{\mathbf{y}}, \quad w_t(\mathbf{x}) = \int_{\mathcal{M}} R_t(\mathbf{x}, \mathbf{y}) \mathrm{d}\mu_{\mathbf{y}}.
$$

By Theorem A.4, we have

$$\begin{aligned}
\|\nabla u\|_{L^2(\mathcal{M})}^2 \leq & 2\|\nabla v\|_{L^2(\mathcal{M})}^2 + 2t^2 \left\|\nabla \left(\frac{r(\mathbf{x})-\bar{r}}{w_t(\mathbf{x})}\right)\right\|_{L^2(\mathcal{M})}^2 \\
\leq & C\left\langle u, L_t u\right\rangle + Ct\|r\|_{L^2(\mathcal{M})}^2 + Ct^2\|\nabla r\|_{L^2(\mathcal{M})}^2 \\
\leq & C\|u\|_{L^2(\mathcal{M})}\|r\|_{L^2(\mathcal{M})} + Ct\|r\|_{L^2(\mathcal{M})}^2 + Ct^2\|\nabla r\|_{L^2(\mathcal{M})}^2 \\
\leq & C\|r\|_{L^2(\mathcal{M})}^2 + Ct^2\|\nabla r\|_{L^2(\mathcal{M})}^2 \\
\leq & C\left(\|r\|_{L^2(\mathcal{M})} + t\|\nabla r\|_{L^2(\mathcal{M})}\right)^2.
\end{aligned}$$

This completes the proof. □

Appendix B. Proof of Theorem 2.3 .

Proof.

The key point is to show that

$$\left|\int_{\mathcal{M}} u(\mathbf{x})\left(r(\mathbf{x})-\bar{r}\right)\mathrm{d}\mu_{\mathbf{x}}\right| \leq C\sqrt{t}\max_{1\leq i\leq d}\left(\|b^i\|_\infty\right)\|u\|_{H^1(\mathcal{M})}. \tag{B.1}$$

Notice that

$$|\bar{r}| = \frac{1}{|\mathcal{M}|}\left|\sum_{i=1}^d \int_{\mathcal{M}}\int_{\partial\mathcal{M}} b^i(\mathbf{y})(x_i-y_i)\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}\mathrm{d}\mathbf{x}\right| \leq C\sqrt{t}\max_{1\leq i\leq d}\left(\|b^i\|_\infty\right).$$

Then it is sufficient to show that

(B.2)

$$\left|\int_{\mathcal{M}} u(\mathbf{x})\left(\int_{\partial\mathcal{M}}\sum_{i=1}^d b^i(\mathbf{y})(x_i-y_i)\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}\right)\mathrm{d}\mu_{\mathbf{x}}\right| \leq C\sqrt{t}\max_{1\leq i\leq d}\left(\|b^i\|_\infty\right)\|u\|_{H^1(\mathcal{M})}.$$

Notice that

$$(x_i-y_i)\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) = 2t\sum_{j=1}^d a_{ij}(\mathbf{x})\nabla_{\mathbf{y}}^j\bar{\bar{R}}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) \tag{B.3}$$

$$= -2t\sum_{j=1}^d a_{ij}(\mathbf{x})\left(\nabla_{\mathbf{x}}^j\bar{\bar{R}}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) + \frac{1}{4t}\sum_{m,n=1}^d \nabla_{\mathbf{x}}^j a^{mn}(\mathbf{x})(x_m-y_m)(x_n-y_n)\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\right)$$

where $\bar{\bar{R}}_t(\mathbf{x},\mathbf{y}) = C_t\bar{\bar{R}}\left(\frac{1}{4t}\sum_{m,n=1}^d (x_m-y_m)a^{mn}(\mathbf{x})(x_n-y_n)\right)$ and $\bar{\bar{R}}(r) = \int_r^\infty \bar{R}(s)\mathrm{d}s$.

By integration by parts, we have

$$\begin{aligned}
&\sum_{i,j=1}^d \int_{\mathcal{M}} u(\mathbf{x})\int_{\partial\mathcal{M}} b^i(\mathbf{y})a_{ij}(\mathbf{x})\nabla_{\mathbf{x}}^j\bar{\bar{R}}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}\mathrm{d}\mathbf{x} \\
=& \sum_{i,j=1}^d \int_{\partial\mathcal{M}}\int_{\partial\mathcal{M}} n_j(\mathbf{x})a_{ij}(\mathbf{x})b^i(\mathbf{y})u(\mathbf{x})\bar{\bar{R}}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{x}}\mathrm{d}\tau_{\mathbf{y}} \\
&- \sum_{i,j=1}^d \int_{\partial\mathcal{M}}\int_{\mathcal{M}} b^i(\mathbf{y})\nabla_{\mathbf{x}}^j[u(\mathbf{x})a_{ij}(\mathbf{x})]\bar{\bar{R}}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{x}\mathrm{d}\tau_{\mathbf{y}}.
\end{aligned} \tag{B.4}$$

For the boundary term,

$$
\begin{aligned}
\text{(B.5)}\quad & \left|\sum_{i,j=1}^{d}\int_{\partial\mathcal{M}}\int_{\partial\mathcal{M}} n_j(\mathbf{x})a_{ij}(\mathbf{x})b^i(\mathbf{y})u(\mathbf{x})\bar{\bar{R}}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{x}}\mathrm{d}\tau_{\mathbf{y}}\right| \\
&\le C\max_{1\le i\le d}\left(\|b^i\|_\infty\right)\int_{\partial\mathcal{M}}\int_{\partial\mathcal{M}}|u(\mathbf{x})|\bar{\bar{R}}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{x}}\mathrm{d}\tau_{\mathbf{y}} \\
&\le C\max_{1\le i\le d}\left(\|b^i\|_\infty\right)\left(\int_{\partial\mathcal{M}}\left(\int_{\partial\mathcal{M}}|u(\mathbf{x})|\bar{\bar{R}}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{x}}\right)^2\mathrm{d}\tau_{\mathbf{y}}\right)^{1/2} \\
&\le C\max_{1\le i\le d}\left(\|b^i\|_\infty\right)\left(\int_{\partial\mathcal{M}}\left(\int_{\partial\mathcal{M}}\bar{\bar{R}}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{x}}\right)\left(\int_{\partial\mathcal{M}}|u(\mathbf{x})|^2\bar{\bar{R}}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{x}}\right)\mathrm{d}\tau_{\mathbf{y}}\right)^{1/2} \\
&\le Ct^{-1/2}\max_{1\le i\le d}\left(\|b^i\|_\infty\right)\|u\|_{L^2(\partial\mathcal{M})}\le Ct^{-1/2}\max_{1\le i\le d}\left(\|b^i\|_\infty\right)\|u\|_{H^1(\mathcal{M})}.
\end{aligned}
$$

The bound of the second term of (B.4) is straightforward. By using the assumption that the coefficients $a_{ij}(\mathbf{x})$ are smooth functions, we have

$$
\begin{aligned}
|\sum_{i,j=1}^{d} b^i(\mathbf{y})\nabla_{\mathbf{x}}^j[u(\mathbf{x})a_{ij}(\mathbf{x})]| &\le \sum_{i,j=1}^{d}|\nabla_{\mathbf{x}}^j u(\mathbf{x})||b^i(\mathbf{y})a_{ij}(\mathbf{x})| + \sum_{i,j=1}^{d}|u(\mathbf{x})||b^i(\mathbf{y})\nabla_{\mathbf{x}}^j a_{ij}(\mathbf{x})| \\
&\le C\max_{1\le i\le d}\left(\|b^i\|_\infty\right)\left(|\nabla u(\mathbf{x})|+|u(\mathbf{x})|\right)
\end{aligned}
$$

where the constant C depends on the curvature of the manifold $\mathcal{M}$.

Then, we have

$$
\begin{aligned}
\text{(B.6)}\quad & \left|\sum_{i,j=1}^{d}\int_{\partial\mathcal{M}}\int_{\mathcal{M}} b^i(\mathbf{y})\nabla_{\mathbf{x}}^j[u(\mathbf{x})a_{ij}(\mathbf{x})]\bar{\bar{R}}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{x}\mathrm{d}\tau_{\mathbf{y}}\right| \\
&\le C\max_{1\le i\le d}\left(\|b^i\|_\infty\right)\int_{\partial\mathcal{M}}\int_{\mathcal{M}}\left(|\nabla u(\mathbf{x})|+|u(\mathbf{x})|\right)\bar{\bar{R}}_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{x}}\mathrm{d}\tau_{\mathbf{y}} \\
&\le C\max_{1\le i\le d}\left(\|b^i\|_\infty\right)\left(\int_{\mathcal{M}}\left(|\nabla u(\mathbf{x})|^2+|u(\mathbf{x})|^2\right)\left(\int_{\partial\mathcal{M}}\bar{\bar{R}}_t(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}\right)\mathrm{d}\mu_{\mathbf{x}}\right)^{1/2} \\
&\le Ct^{-1/4}\max_{1\le i\le d}\left(\|b^i\|_\infty\right)\|u\|_{H^1(\mathcal{M})}.
\end{aligned}
$$

and

(B.7)

$$
\begin{aligned}
&\left|\int_{\mathcal{M}} u(\mathbf{x})\left(\int_{\partial\mathcal{M}}\sum_{i,j,m,n=1}^{d} b^i(\mathbf{y})a_{ij}(\mathbf{x})\nabla_{\mathbf{x}}^j a^{mn}(\mathbf{x})(x_m-y_m)(x_n-y_n)\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}\right)\mathrm{d}\mathbf{x}\right| \\
\le & Ct\int_{\mathcal{M}}|u(\mathbf{x})|\left(\int_{\partial\mathcal{M}}\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}\right)\mathrm{d}\mathbf{x}\le Ct^{3/4}\|u\|_{L^2}.
\end{aligned}
$$

Then, the inequality (B.2) is obtained from (B.3), (B.4), (B.5), (B.6) and (B.7). Now, using Theorem A.3, we have

$$
\text{(B.8)}\qquad \|u\|_{L^2(\mathcal{M})}^2\le C\langle u, L_t u\rangle\le C\sqrt{t}\max_i\left(\|b^i\|_\infty\right)\|u\|_{H^1(\mathcal{M})}.
$$

Note $r(\mathbf{x}) = \sum_{i=1}^d \int_{\partial\mathcal{M}} b^i(\mathbf{y})(x_i - y_i)\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}$. A direct calculation gives us that

$$\|r(\mathbf{x})\|_{L^2(\mathcal{M})} \leq Ct^{1/4} \max_{1\leq i\leq d}\left(\|b^i\|_\infty\right), \text{ and} \tag{B.9}$$

$$\|\nabla r(\mathbf{x})\|_{L^2(\mathcal{M})} \leq Ct^{-1/4} \max_{1\leq i\leq d}\left(\|b^i\|_\infty\right). \tag{B.10}$$

The integral equation $-L_t u = r - \bar{r}$ gives that

$$u(\mathbf{x}) = v(\mathbf{x}) + \frac{t}{w_t(\mathbf{x})}\left(r(\mathbf{x}) - \bar{r}\right) \tag{B.11}$$

where

$$v(\mathbf{x}) = \frac{1}{w_t(\mathbf{x})}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}, \quad w_t(\mathbf{x}) = \int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}. \tag{B.12}$$

By Theorem A.4, we have

$$\begin{aligned}
&\|\nabla u\|_{L^2(\mathcal{M})}^2 \\
&\leq 2\|\nabla v\|_{L^2(\mathcal{M})}^2 + 2t^2\left\|\nabla\left(\frac{r(\mathbf{x}) - \bar{r}}{w_t(\mathbf{x})}\right)\right\|_{L^2(\mathcal{M})}^2 \\
&\leq C\langle u, L_t u\rangle + Ct\|r\|_{L^2(\mathcal{M})}^2 + Ct^2\|\nabla r\|_{L^2(\mathcal{M})}^2 \\
&\leq C\sqrt{t}\max_{1\leq i\leq d}\left(\|b^i\|_\infty\right)\|u\|_{H^1(\mathcal{M})} + Ct\|r\|_{L^2(\mathcal{M})}^2 + Ct^2\|\nabla r\|_{L^2(\mathcal{M})}^2 \\
&\leq C\max_{1\leq i\leq d}\left(\|b^i\|_\infty\right)\left(\sqrt{t}\|u\|_{H^1(\mathcal{M})} + Ct^{3/2}\right).
\end{aligned} \tag{B.13}$$

Using (B.8) and (B.13), we have

$$\|u\|_{H^1(\mathcal{M})}^2 \leq C\max_{1\leq i\leq d}\left(\|b^i\|_\infty\right)\left(\sqrt{t}\|u\|_{H^1(\mathcal{M})} + Ct^{3/2}\right), \tag{B.14}$$

which proves the theorem. □

Appendix C. Derivation of Eq. (3.19)**.**

Denote $\mathbf{A}(\mathbf{x}) = (a_{ij}(\mathbf{x})) \in \mathbb{R}^{d\times d}$. Let $\mathbf{X} = [\partial_1\Phi, \partial_2\Phi, \cdots, \partial_m\Phi]$ be an orthonormal basis of the tangent space $\mathcal{T}_{\mathbf{x}}(\mathcal{M})$ at $\mathbf{x}$ and $\mathbf{Y}$ be the orthogonal completion of $\mathbf{X}$ in $\mathbb{R}^d$. Then we have

$$\mathbf{AX} = \mathbf{XC}, \quad \mathbf{AY} = \mathbf{YD},$$

since the tangent space $\mathcal{T}_{\mathbf{x}}(\mathcal{M})$ is a invariant subspace of $\mathbf{A}(\mathbf{x})$. This gives a decomposition of $\mathbf{A}$

$$\mathbf{A} = \mathbf{P}\begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}\end{bmatrix}\mathbf{P}^{-1}, \qquad \mathbf{P} = [\mathbf{X}, \mathbf{Y}], \quad \mathbf{P}^{-1} = \begin{bmatrix} (\mathbf{X^TX})^{-1}\mathbf{X^T} \\ (\mathbf{Y^TY})^{-1}\mathbf{Y^T}\end{bmatrix} \tag{C.1}$$

Using these notations, we have

$$\begin{aligned}
&D_i a^{mn}(\mathbf{y}) a_{km}(\mathbf{y})(\partial_{i'}\Phi^k g^{i'j'}\partial_{j'}\Phi^n)\\
=&\text{trace}(D_i(\mathbf{A}^{-1})\mathbf{A}\mathbf{X}(\mathbf{X^T X})^{-1}\mathbf{X^T})\\
=&\text{trace}(D_i(\mathbf{A}^{-1})\mathbf{X}\mathbf{C}(\mathbf{X^T X})^{-1}\mathbf{X^T})\\
=&\text{trace}\left(\mathbf{P}D_i\left(\begin{bmatrix}\mathbf{C}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{D}^{-1}\end{bmatrix}\right)\mathbf{P}^{-1}\mathbf{X}\mathbf{C}(\mathbf{X^T X})^{-1}\mathbf{X^T}\right)\\
&+\text{trace}\left(D_i(\mathbf{P})\left(\begin{bmatrix}\mathbf{C}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{D}^{-1}\end{bmatrix}\right)\mathbf{P}^{-1}\mathbf{X}\mathbf{C}(\mathbf{X^T X})^{-1}\mathbf{X^T}\right)\\
&+\text{trace}\left(\mathbf{P}\left(\begin{bmatrix}\mathbf{C}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{D}^{-1}\end{bmatrix}\right)D_i(\mathbf{P}^{-1})\mathbf{X}\mathbf{C}(\mathbf{X^T X})^{-1}\mathbf{X^T}\right).
\end{aligned}$$

Then, we calculate three terms one by one.

$$\text{trace}\left(\mathbf{P}D_i\left(\begin{bmatrix}\mathbf{C}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{D}^{-1}\end{bmatrix}\right)\mathbf{P}^{-1}\mathbf{X}\mathbf{C}(\mathbf{X^T X})^{-1}\mathbf{X^T}\right) = \text{trace}\left(D_i(\mathbf{C}^{-1})\mathbf{C}\right),$$

$$\text{trace}\left(D_i(\mathbf{P})\left(\begin{bmatrix}\mathbf{C}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{D}^{-1}\end{bmatrix}\right)\mathbf{P}^{-1}\mathbf{X}\mathbf{C}(\mathbf{X^T X})^{-1}\mathbf{X^T}\right) = \text{trace}\left(D_i(\mathbf{X})(\mathbf{X^T X})^{-1}\mathbf{X^T}\right),$$

$$\text{trace}\left(\mathbf{P}\left(\begin{bmatrix}\mathbf{C}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{D}^{-1}\end{bmatrix}\right)D_i(\mathbf{P}^{-1})\mathbf{X}\mathbf{C}(\mathbf{X^T X})^{-1}\mathbf{X^T}\right) = \text{trace}\left(\mathbf{X}D_i((\mathbf{X^T X})^{-1}\mathbf{X^T})\right).$$

Also notice that

$$\begin{aligned}
&\text{trace}\left(D_i(\mathbf{X})(\mathbf{X^T X})^{-1}\mathbf{X^T}\right) + \text{trace}\left(\mathbf{X}D_i((\mathbf{X^T X})^{-1}\mathbf{X^T})\right)\\
=&D_i\left(\text{trace}\left((\mathbf{X^T X})^{-1}\mathbf{X^T X}\right)\right) = 0.
\end{aligned}$$

Combining all the calculations together, we get

$$D_i a^{mn}(\mathbf{y}) a_{km}(\mathbf{y})(\partial_{i'}\Phi^k g^{i'j'}\partial_{j'}\Phi^n) = \text{trace}\left(D_i(\mathbf{C}^{-1})\mathbf{C}\right) = \frac{1}{\sqrt{\det(\mathbf{C})}}D_i(\sqrt{\det(\mathbf{C})}).$$

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