

# Generalization of the Weighted Nonlocal Laplacian in Low Dimensional Manifold Model

Zuoqiang Shi · Stanley Osher · Wei Zhu

Received: date / Accepted: date

**Abstract** In this paper we use the idea of the weighted nonlocal Laplacian [20] to deal with the constraints in the low dimensional manifold model [15]. In the original LDMM, the constraints are enforced by the point integral method. The point integral method provides a correct way to deal with the constraints, however it is not very efficient due to the fact that the symmetry of the original Laplace-Beltrami operator is destroyed. WNLL provides another way to enforce the constraints in LDMM. In WNLL, the discretized system is symmetric and sparse and hence it can be solved very fast. Our experimental results show that the computational cost is reduced significantly with the help of WNLL. Moreover, the results in image inpainting and denoising are also better than the original LDMM and competitive with state-of-the-art methods.

**Keywords** Weighted nonlocal Laplacian · low dimensional manifold model · nonlocal methods · point cloud

**Mathematics Subject Classification (2000)** 65D05 · 65D25 · 41A05

---

Research supported by DOE-SC0013838 and NSF DMS-1118971. Z. Shi was partially supported by NSFC Grant 11371220, 11671005.

Zuoqiang Shi  
Yau Mathematical Sciences Center, Tsinghua University, Beijing, China, 100084.  
E-mail: zqshi@mail.tsinghua.edu.cn

Stanley Osher  
Department of Mathematics, University of California, Los Angeles, CA 90095, USA.  
E-mail: sjo@math.ucla.edu

Wei Zhu  
Department of Mathematics, University of California, Los Angeles, CA 90095, USA.  
E-mail: weizhu731@math.ucla.edu

## 1 Introduction

In the past decade patch-based methods have achieved great success in image denoising, inpainting and other image processing problems [8, 4, 3, 6, 11, 25, 10, 28, 9, 2, 32]. Among different patch-based nonlocal methods, the manifold model is attracting more and more attention, along with the development of manifold learning algorithms. The basic assumption in the manifold model is that the patches concentrate around a low dimensional smooth manifold. This assumption is verified in studies in image processing and computer vision [17–19, 12, 5, 16].

Recently the low dimensional manifold model (LDMM) was proposed for image processing [15]. This model explicitly use the dimension of the patch manifold as a regularization to reconstruct the image. One way to exploit the low dimensional structure is to locally study the patch manifold. This idea leads to the methods of multiple subspaces which approximate the patch manifold locally by linear spaces [1, 27, 31, 30, 29]. In LDMM, with the help of an elegant formula in differential geometry, we formulate an optimization problem of minimizing the  $H^1$  norm of the coordinate functions of the patch manifold. The key step in the algorithm is to solve a Laplace-Beltrami equation on the patch set. This equation is solved by the point integral method [13, 14, 21, 22]. LDMM has been shown to have very good performance in image processing, especially in inpainting problems. On the other hand, the computational cost of the original LDMM in [15] is relatively high. There are mainly two difficulties which increase the computational load. First, the linear systems to be solved from the point integral method are usually not symmetric and can not be solved very efficiently. Secondly, in LDMM, the image is recovered iteratively. To get a good reconstruction, the number of iterations is usually over 100. In this paper we use the idea of the weighted nonlocal Laplacian to replace the point integral method to enforce the constraints in LDMM. It turns out that the computational cost is reduced significantly and the results in image inpainting and denoising become even better.

The weighted nonlocal Laplacian (WNLL) [20] is a recently developed method to compute smooth interpolation of a function partially sampled on a point cloud. The key idea in the weighted nonlocal Laplacian is to separate the energy function into two parts: one in the labeled set and the other in the unlabeled set. In WNLL, we put a weight in front of the labeled set to correctly enforce the sampling constraints.

$$\min_u \sum_{\mathbf{x} \in P \setminus S} \left( \sum_{\mathbf{y} \in P} w(\mathbf{x}, \mathbf{y})(u(\mathbf{x}) - u(\mathbf{y}))^2 \right) + \frac{|P|}{|S|} \sum_{\mathbf{x} \in S} \left( \sum_{\mathbf{y} \in P} w(\mathbf{x}, \mathbf{y})(u(\mathbf{x}) - u(\mathbf{y}))^2 \right), \quad (1.1)$$

with the constraint

$$u(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in S,$$

where  $P$  is the point cloud and  $S$  is the labeled set,  $w(\mathbf{x}, \mathbf{y})$  is a given weight function and  $g$  is the given values on  $S$ . The equation in WNLL is symmetric and positive definite, which is much easier to solve than that in the point integral method. In this paper, we generalize the idea of the weighted nonlocal Laplacian to image denoising and get comparable results as the well-known BM3D method [8].

Moreover, we also use the semi-local patches to construct the weight matrix. The semi-local patches are combinations of nonlocal patches and local coordinates,

$$(\bar{\mathcal{P}}\mathbf{u})(x) = [(\mathcal{P}\mathbf{u})(x), \lambda x],$$

where  $\mathbf{u} \in \mathbb{R}^{m \times n}$  is an image,  $x \in \bar{\Omega} = \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$  is the index of the pixel,  $\mathcal{P}\mathbf{u}(x)$  is the patch at  $x$  to be defined in Section 2. The weight  $\lambda$  is used to get different locality in the semi-local patch. The semi-local patches have been applied to many image processing problems [17, 26, 23, 24, 7]. They provide a way to balance the local and nonlocal effects. With a proper choice of  $\lambda$ , semi-local patches could significantly decrease the number of iterations in LDMM and give better results.

Using the semi-local patches and WNLL, we can reduce the number of iterations in LDMM from 100 to 10. In each step, the linear system is easier to solve. Thus, the total computational time is reduced tremendously. Also the results are better than those in [15] in both image inpainting and denoising.

The rest of the paper is organized as follows. For the completeness of this paper, we briefly introduce LDMM and WNLL in Sections 2 and 3 respectively. The semi-local patches are discussed in Section 4. We apply the new LDMM method to image inpainting and denoising in Section 5 and Section 6. Finally, we discuss the weighted nonlocal Laplacian with general constraints and some future works in Section 7.

## 2 Low Dimensional Manifold Model

Recently, the low dimensional manifold model (LDMM) was proposed for image processing, which gives very good results especially for subsampled image reconstruction [15]. The main idea in LDMM is using the dimension of the patch manifold associated with the image as a regularization to recover the image such that the associated patch manifold has the lowest dimension.

Consider a discrete image  $\mathbf{u} \in \mathbb{R}^{m \times n}$ , for any  $x \in \bar{\Omega} = \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ , we define a patch  $(\mathcal{P}\mathbf{u})(x)$  as a 2D rectangle of size  $s_1 \times s_2$  of the original image  $\mathbf{u}$ , and the pixel  $x$  is the top-left corner of that rectangle. The collection of all patches is called the patch set of image  $\mathbf{u}$  and is denoted as  $P(\mathbf{u})$ :

$$P(\mathbf{u}) = \{(\mathcal{P}\mathbf{u})(x) : x \in \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}\} \subset \mathbb{R}^d, \quad d = s_1 \times s_2. \quad (2.1)$$

For a given image  $\mathbf{u}$ , the patch set  $P(\mathbf{u})$  give a point cloud in  $\mathbb{R}^d$ . It is observed that this point cloud is usually close to a smooth manifold embedded in  $\mathbb{R}^d$ . This underlying smooth manifold is called the *patch manifold* associated with  $\mathbf{u}$ , denoted as  $\mathcal{M}(\mathbf{u})$ . Many studies have revealed that for many natural images, the dimension of the patch manifold is usually very low [17–19, 12, 5, 16]. Thus we can use the dimension of the patch manifold as a regularization to recover the image:

$$\min_{\substack{\mathbf{u} \in \mathbb{R}^{m \times n}, \\ \mathcal{M} \subset \mathbb{R}^d}} \dim(\mathcal{M}), \quad \text{subject to: } \mathbf{b} = \Phi \mathbf{u} + \varepsilon, \quad P(\mathbf{u}) \subset \mathcal{M}, \quad (2.2)$$

where  $\dim(\mathcal{M})$  is the dimension of the manifold  $\mathcal{M}$ ,  $\mathbf{b}$  is the observation,  $\Phi$  is a measurement operator,  $\varepsilon$  represents the noise perturbation.

In LDMM, the dimension of the patch manifold is calculated using the following formula:

$$\dim(\mathcal{M}) = \sum_{j=1}^d \|\nabla_{\mathcal{M}} \alpha_j(\mathbf{x})\|^2.$$

As a result, the optimization problem (2.2) can be rewritten as

$$\min_{\substack{\mathbf{u} \in \mathbb{R}^{m \times n}, \\ \mathcal{M} \subset \mathbb{R}^d}} \sum_{i=1}^d \|\nabla_{\mathcal{M}} \alpha_i\|_{L^2(\mathcal{M})}^2 + \mu \|\mathbf{b} - \Phi \mathbf{u}\|_2^2, \quad \text{subject to: } P(\mathbf{u}) \subset \mathcal{M}. \quad (2.3)$$

where  $\mu$  is a parameter in the penalty term.

The above optimization problem is solved by Bregman iterations. The key step is solving a Poisson equation on a point cloud. In [15], this equation is solved by the point integral method [13, 14, 21, 22]. LDMM using the point integral method has been shown to achieve good performances, especially in image inpainting [15]. However, the computational cost of the original LDMM in [15] is relatively high. First, the linear system arising from the point integral method is usually not symmetric. It is not efficient to solve numerically. Secondly, in LDMM, we recover the image iteratively, based on completely nonlocal patches. We need many iterations, typically over 100, for the algorithm to converge to a satisfactory reconstruction. To reduce the computational cost of LDMM, we propose two procedures in this paper. First, we use the weighted nonlocal Laplacian [20] to replace the point integral method. The linear system given by WNLL is symmetric and positive definite which is much easier to solve. Secondly, we use semi-local patches instead of nonlocal patches. Our numerical experiments show that semi-local patches could reduce the number of iterations tremendously, from over 100 to around 10.

### 3 Weighted Nonlocal Laplacian

The weighted nonlocal Laplacian (WNLL) is a method to compute smooth interpolation on a point cloud, which was recently developed in [20]. The key idea in the weighted nonlocal Laplacian is to separate the energy function in the graph Laplacian into two parts: one in the labeled set and the other in the unlabeled set. When the sample rate is low, the term in the unlabeled set overwhelms the other term, sacrificing the continuity in the labeled set. To ensure the continuity, one simple method is to add a weight to balance these two terms. One natural choice of the weight is the inverse of the sample rate, which gives the following optimization problem

$$\min_u \sum_{\mathbf{x} \in P \setminus S} \left( \sum_{\mathbf{y} \in P} w(\mathbf{x}, \mathbf{y})(u(\mathbf{x}) - u(\mathbf{y}))^2 \right) + \frac{|P|}{|S|} \sum_{\mathbf{x} \in S} \left( \sum_{\mathbf{y} \in P} w(\mathbf{x}, \mathbf{y})(u(\mathbf{x}) - u(\mathbf{y}))^2 \right), \quad (3.1)$$

with the constraint

$$u(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in S,$$

where  $P$  is the point cloud and  $S$  is the labeled set,  $w(\mathbf{x}, \mathbf{y})$  is a given weight function and  $g$  is the given values on  $S$ .

The corresponding Euler-Lagrange equation of (3.1) is a linear system

$$\begin{aligned} \sum_{\mathbf{y} \in P} (w(\mathbf{x}, \mathbf{y}) + w(\mathbf{y}, \mathbf{x}))(u(\mathbf{x}) - u(\mathbf{y})) & \quad (3.2) \\ + \left( \frac{|P|}{|S|} - 1 \right) \sum_{\mathbf{y} \in S} w(\mathbf{y}, \mathbf{x})(u(\mathbf{x}) - g(\mathbf{y})) = 0, & \quad \mathbf{x} \in P \setminus S, \\ u(\mathbf{x}) = g(\mathbf{x}), & \quad \mathbf{x} \in S. \end{aligned}$$

This linear system is symmetric and positive definite. In addition, in WNLL, a large positive term is added to the diagonal of the coefficient matrix, which makes the conjugate gradient method converge in fewer steps.

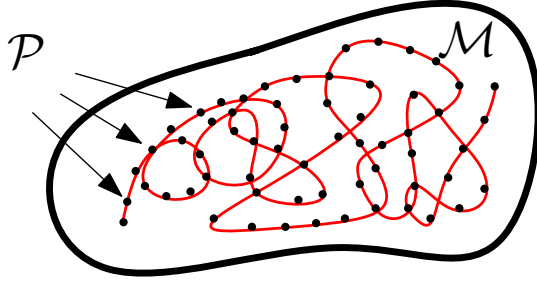
Besides the variational approach, we can derive the weighted nonlocal Laplacian by modifying the point integral method.

If we use the point integral method [13, 14, 21, 22] to solve the interpolation problem on point cloud, the equations are

$$\sum_{\mathbf{y} \in P} R_t(\mathbf{x}, \mathbf{y})(u(\mathbf{x}) - u(\mathbf{y})) + \frac{2}{\mu} \sum_{\mathbf{y} \in S} \bar{R}_t(\mathbf{x}, \mathbf{y})(u(\mathbf{y}) - g(\mathbf{y})) = 0, \quad (3.3)$$

where  $0 < \mu \ll 1$  is a small parameter,  $R_t(\mathbf{x}, \mathbf{y}) = R(-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t})$ ,  $\bar{R}_t(\mathbf{x}, \mathbf{y}) = \bar{R}(-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t})$  and  $\frac{d}{ds} \bar{R}(s) = -R(s)$ . If  $R(s) = \exp(-s)$ ,  $\bar{R} = R$  and  $R_t(\mathbf{x}, \mathbf{y})$  becomes a Gaussian weight function.

Comparing above equation (3.3) with the equation in WNLL, (3.2), the only difference is that, in the second summation,  $u(\mathbf{y})$  in PIM is replaced



**Fig. 1** Diagram of patch set  $\mathcal{P}$  (black points), local 2D parameterization (red curve) and global patch manifold parameterization.

by  $u(\mathbf{x})$  in WNLL. The difference is very small, however, the consequence is significant. Intuitively, this modification also makes sense. Since  $u$  is a smooth function,  $|u(x) - u(y)| = O|x - y|$  is small due to the small support of the weight function  $w(\mathbf{x}, \mathbf{y})$ .

#### 4 Semi-local Patches

The semi-local patches are obtained by adding local coordinates to the nonlocal patches in Section 2 with a weight  $\lambda$ , i.e.

$$(\bar{\mathcal{P}}\mathbf{u})(x) = [(\mathcal{P}\mathbf{u})(x), \lambda x] \in \mathbb{R}^{d+2}, \quad (4.1)$$

and the semi-local patch set is

$$\bar{\mathcal{P}}\mathbf{u} = \{(\bar{\mathcal{P}}\mathbf{u})(x) : x \in \bar{\Omega} = \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}\}. \quad (4.2)$$

where  $\mathbf{u} \in \mathbb{R}^{m \times n}$  is an image,  $x \in \bar{\Omega} = \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$  is the index of the pixel. The weight  $\lambda$  is used to get different locality in the semi-local patch. If  $\lambda = 0$ , the semi-local patch is same as the nonlocal patch. If  $\lambda \rightarrow \infty$ , the patches are completely determined by local coordinates.

The semi-local patches have been successfully applied in many image processing problems [17, 26, 23, 24, 7]. In the geometrical point of view, if  $\lambda \rightarrow \infty$ , the patch set  $\bar{\mathcal{P}}(f)$  is parametrized by a local 2D coordinate,  $x \mapsto \bar{\mathcal{P}}(\mathbf{u})(x)$ . However, this parameterization is globally not injective and typically leads to high curvature variations and self-intersections. With this parametrization, the dimension of the manifold is very low, while the regularity is poor. If  $\lambda = 0$ , the underlying patch manifold may have higher dimension, however the manifold becomes smoother. Fig. 1 gives a diagram which shows these two different parameterizations. The idea in this paper is to find a good compromise between the dimension and the regularity of the patch manifold by choosing a proper weight  $\lambda$ . With a proper  $\lambda$ , our experiments show that LDMM can be accelerated significantly.

## 5 Image Inpainting

In this section, we apply LDMM with semi-local patches to image inpainting problems. In image inpainting we want to recover the image  $\mathbf{u} \in \mathbb{R}^{m \times n}$  from its subsamples  $\mathbf{u}|_{\Omega}$ , where  $\Omega \subset \bar{\Omega} = \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$  is the subsampled domain. In this paper we consider the noise free case.

The key step in LDMM is to solve the following optimization problem:

$$\begin{aligned} \min_{\mathbf{u} \in \mathbb{R}^{m \times n}} \quad & \sum_{i=1}^d \|\nabla_{\mathcal{M}} \alpha_i\|_{L^2(\mathcal{M})}^2, \\ \text{subject to:} \quad & \alpha_i(\bar{\mathcal{P}}(\mathbf{u}^k)(x)) = (\mathcal{P}_i \mathbf{u})(x), \quad x \in \bar{\Omega}, \quad i = 1, \dots, d, \\ & \mathbf{u}(x) = \mathbf{b}(x), \quad x \in \Omega \subset \bar{\Omega}. \end{aligned} \quad (5.1)$$

where  $\bar{\mathcal{P}}(\mathbf{u}^k)(x)$  is the semi-local patch of  $\mathbf{u}^k$  at  $x$ ,  $(\mathcal{P}_i \mathbf{u})(x)$  is the  $i$ th element of patch  $(\mathcal{P} \mathbf{u})(x)$ .

Based on the weighted nonlocal Laplacian, to get a correct solution, we need to put an extra weight on the sampled set. Notice that the sample set is different for different  $\alpha_i$ . Applying WNLL, we get the following discretized optimization problem:

$$\begin{aligned} \min_{\mathbf{u} \in \mathbb{R}^{m \times n}} \quad & \sum_{i=1}^d \left( \sum_{x \in \bar{\Omega} \setminus \Omega_i} \sum_{y \in \bar{\Omega}} \bar{w}(x, y) ((\mathcal{P}_i \mathbf{u})(x) - (\mathcal{P}_i \mathbf{u})(y))^2 \right. \\ & \left. + \frac{mn}{|\Omega|} \sum_{x \in \Omega_i} \sum_{y \in \bar{\Omega}} \bar{w}(x, y) ((\mathcal{P}_i \mathbf{u})(x) - (\mathcal{P}_i \mathbf{u})(y))^2 \right), \end{aligned} \quad (5.2)$$

with the constraints

$$\mathbf{u}(x) = \mathbf{b}(x), \quad x \in \Omega \subset \bar{\Omega}.$$

$\Omega_i = \{x \in \bar{\Omega} : (\mathcal{P}_i \mathbf{u})(x) \text{ is sampled}\}$  is the sampled set of  $\alpha_i$  and

$$\bar{w}(x, y) = w((\bar{\mathcal{P}} \mathbf{u})(x), (\bar{\mathcal{P}} \mathbf{u})(y)), \quad (5.3)$$

where  $w$  is the given weight function.

Using a standard variational approach, we get the Euler-Lagrange equation of the above optimization problem:

$$\begin{aligned} \sum_{i=1}^d \mathcal{P}_i^* \left( \sum_{y \in \bar{\Omega}} (\bar{w}(x, y) + \bar{w}(y, x)) ((\mathcal{P}_i \mathbf{u})(x) - (\mathcal{P}_i \mathbf{u})(y)) \right) \\ + \beta \sum_{i=1}^d \mathcal{P}_i^* \left( \sum_{y \in \Omega_i} \bar{w}(y, x) ((\mathcal{P}_i \mathbf{u})(x) - (\mathcal{P}_i \mathbf{u})(y)) \right) = 0, \quad x \in \bar{\Omega} \setminus \Omega, \\ \mathbf{u}(x) = \mathbf{b}(x), \quad x \in \Omega, \end{aligned} \quad (5.4)$$

**Algorithm 1** Image Inpainting**Require:** A subsample image  $\mathbf{u}|_{\Omega}$ .**Ensure:** A recovered image  $\mathbf{u}$ .Generate initial image  $\mathbf{u}^0$ .**while** not converge **do**

1. Generate semi-local patch set  $\bar{P}(\mathbf{u}^k)$  from current image  $\mathbf{u}^k$  with the weight  $\lambda$  in the semi-local patch.
2. Compute the weight function

$$\bar{w}^k(x, y) = w((\bar{\mathcal{P}}\mathbf{u}^k)(x), (\bar{\mathcal{P}}\mathbf{u}^k)(y)), \quad x, y \in \bar{\Omega}.$$

3. Update the image  $\mathbf{u}^{k+1}$  by solving equation (5.4) with  $\bar{w}^k(x, y)$ .
4.  $k \leftarrow k + 1$ .

**end while** $\mathbf{u} = \mathbf{u}^k$ .

where  $\mathcal{P}_i^*$  is the adjoint operator of  $\mathcal{P}_i$  and  $\beta = \frac{mn}{|\Omega|} - 1$ .

Now, we get an algorithm for image inpainting, Algorithm 1.

Next, we will apply Algorithm 1 to some image inpainting problems. In our calculations below, the weight  $w(x, y)$  is calculated as:

$$w(\mathbf{x}, \mathbf{y}) = \exp\left(-\frac{\|\mathbf{x} - \mathbf{y}\|^2}{\sigma(\mathbf{x})^2}\right). \quad (5.5)$$

where  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{d+2}$  are semi-local patches,  $\sigma(\mathbf{x})$  is chosen to be the distance between  $\mathbf{x}$  and its 20th nearest neighbor, To make the weight matrix sparse, we truncate the weight to the 50 nearest neighbors. The patch size is  $10 \times 10$ . For each patch, the nearest neighbors are obtained by using an approximate nearest neighbor (ANN) search algorithm. We use a k-d tree approach as well as an ANN search algorithm to reduce the computational cost. The linear system in Algorithm 1 is solved by the conjugate gradient method.

In the semi-local patches, the local coordinate is normalized to have the same amplitude as the image intensity,

$$(\bar{\mathcal{P}}\mathbf{u})(\bar{x}) = [(\mathcal{P}\mathbf{u})(x), \lambda\bar{x}]$$

with

$$\bar{x} = \left(\frac{x_1 \|f|_{\Omega}\|_{\infty}}{m}, \frac{x_2 \|f|_{\Omega}\|_{\infty}}{n}\right),$$

$x = (x_1, x_2)$  and  $m, n$  are the sizes of the image.

Large  $\lambda$  makes it easier to compute the weight function  $w$ , because the searching of the nearest neighbors is confined to a local area. On the other hand, small  $\lambda$  gives more accurate results, since more global information is used. Based on this observation, in the computation with 10% subsamples, we gradually reduce  $\lambda$  in the iteration, more specially,  $\lambda$  is determined in the following way,

$$\lambda^{k+1} = \max(\lambda^k - 1, 3),$$



and the initial  $\lambda$  is set to be 10. In the computation with 5% subsamples, we fix  $\lambda$  to be 3, since more global information is required to get good reconstruction with lower sample rate. In all the computations, the number of iterations is fixed to be 10.

PSNR defined as follows is used to measure the accuracy of the results

$$\text{PSNR}(f, f^*) = -20 \log_{10}(\|f - f^*\|/255), \quad (5.6)$$

where  $f^*$  is the ground truth.

Fig. 2 displays the results with 10% subsamples. GL and WNLL are the methods presented in [20]. In all the examples, LDMM gives the best results. Compared with WNLL, the improvement in PSNR is not significant. However, we can see more differences visually. For instance, there sometimes exists spurious structures in the results of WNLL. As shown in the reconstruction of WNLL in Fig. 3, there is a white line near the garlic that does not exist in the original image. In the result of LDMM, this white line is removed. We can also see this phenomenon in the other examples: for instance, the right-top corner in the image of boat, the sky near the roof in the image of castle.

In the results with 5% subsamples, the improvement of LDMM is more significant as shown in Fig. 4. We also study the effect of the weight  $\lambda$  in the semi-local patches. In Fig. 5, we show the reconstructions from 5% subsamples with different  $\lambda$ ,  $\lambda = 3$  and 10 respectively. It turns out that different images have different preferences. As shown in Fig. 5, the image of Barbara prefers smaller  $\lambda$ , while the image of boat likes larger  $\lambda$ . This also makes sense intuitively. There is a large area with repeated patterns in the image of Barbara. This suggests the existence of a global structure, and we need more global information to reconstruct the image. Contrary to the image of Barbara, there is no repeated pattern in the image of boat. Too much global information may introduce some artificially repeated patterns which would corrupt the image. So far, we do not have a very precise procedure to select  $\lambda$ . In principle, reconstruction of repeated patterns prefers small  $\lambda$ , while reconstruction of local structures likes large  $\lambda$ .

With the weighted nonlocal Laplacian and the semi-local patches, we reduce the computational time of LDMM significantly. For instance, to reconstruct the image of Barbara,  $256 \times 256$  from 10% subsamples, the original LDMM in [15] takes about 20 minutes, while the new algorithm takes about 5 minutes. In addition, we found that in the new algorithm, over 60% of computational time is spent on assembling the coefficient matrix in (5.4). There are many loops in assembling the coefficient matrix and our code is written in matlab. We may be able to reduce the computational time further after optimizing this part of the code.

In addition to the computational time being reduced, the results are also better than those of the LDMM with nonlocal patches in [15]. From a geometrical point of view, the dimension becomes smaller in semi-local patches as explained in Section 4.

At the end of this section, we compare LDMM with two recent methods, EBI [9] and PLE [29]. Table 1 gives the inpainting results in PSNR. LDMM



Fig. 2 Results of subsample image restoration from 10% subsamples.

considerably outperforms EBI and PLE when the sample rate is low. The improvement of PSNR is 2-3 dB with 5% samples and about 2 dB with 10% samples. When the sample rate is increased to 20%, for the images of Boat and Pepper which are dominated by cartoons, PLE gives best results in PSNR. However, the improvement is not very significant ( $< 0.5$  dB). As for the image of Barbara, which is rich in textures, LDMM still gains about 3 dB over EBI and PLE.

Fig. 6-8 show the results of different methods on the images of Barbara and Pepper with different sample rate. With 20% sample rate, on the image of

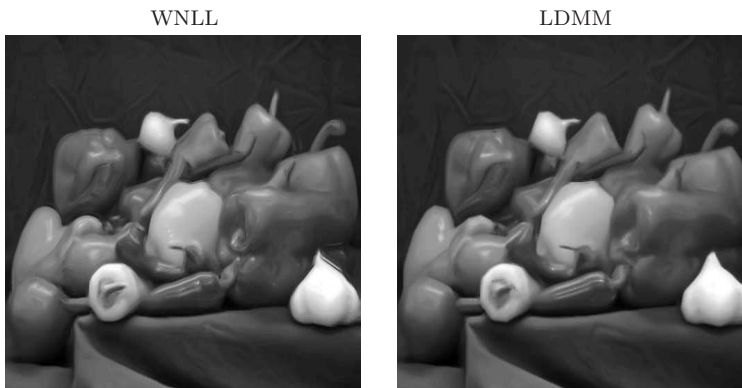


Fig. 3 Reconstruction of the image of peppers in WNLL and LDMM.

	5%			10%			20%		
	EBI	PLE	LDMM	EBI	PLE	LDMM	EBI	PLE	LDMM
Barbara	20.66	19.72	<b>23.38</b>	23.91	22.48	<b>25.97</b>	25.29	25.70	<b>28.61</b>
Boat	20.55	20.54	<b>23.17</b>	23.74	24.19	<b>25.66</b>	26.02	<b>28.14</b>	27.97
Pepper	27.58	25.87	<b>29.56</b>	29.79	30.63	<b>31.66</b>	32.23	<b>35.06</b>	34.63
Castle	20.54	20.78	<b>22.10</b>	22.05	22.93	<b>23.55</b>	23.72	25.15	<b>25.88</b>
Man	22.47	21.98	<b>24.55</b>	24.83	25.64	<b>26.49</b>	25.99	27.91	<b>28.30</b>

Table 1 PSNR of EBI, PLE, LDMM with 5%, 10% and 20% sample rate.

Pepper, all methods give good recovery. Visually there are almost no difference. On the image of Barbara, PLE has some artificial oscillations in the texture part. EBI and LDMM give much better results. When the sample rate is decreased to 10%, PLE can not restore textures very well. LDMM and EBI give better restorations in texture part, while the restoration of LDMM is smoother than that of EBI. If the sample rate is further decreased to 5%. PLE fails to give good reconstructions on both of Barbara and Pepper. LDMM and EBI still give reasonable results. In this case, we can clearly see that LDMM gives sharper reconstructions of edges than EBI. However, we have to say that the reconstruction of EBI in textures is very impressive, although there are some artificial oscillations when the sample rate is low.

## 6 Image Denoising

In this section, we apply the LDMM and WNLL to image denoising. Compared with the image inpainting, the constraints in denoising are different,

$$\mathbf{b}(x) = \mathbf{u}(x) + \varepsilon(x), \quad x \in \overline{\Omega} = \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}, \quad (6.1)$$

where  $\mathbf{u} \in \mathbb{R}^{m \times n}$  is the original image,  $\mathbf{b} \in \mathbb{R}^{m \times n}$  is the noised image,  $\varepsilon$  is the noise. In this paper, we assume  $\varepsilon$  is the Gaussian noise.

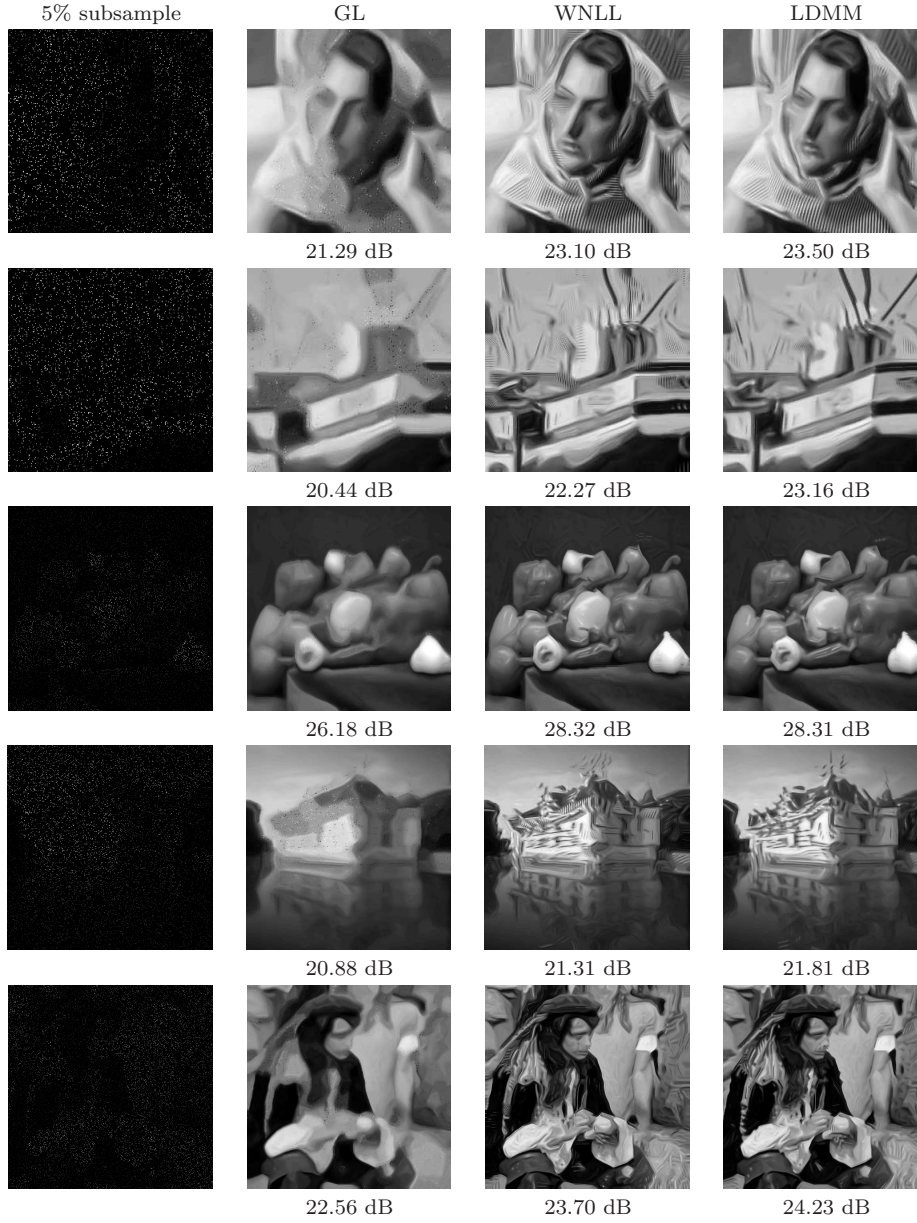
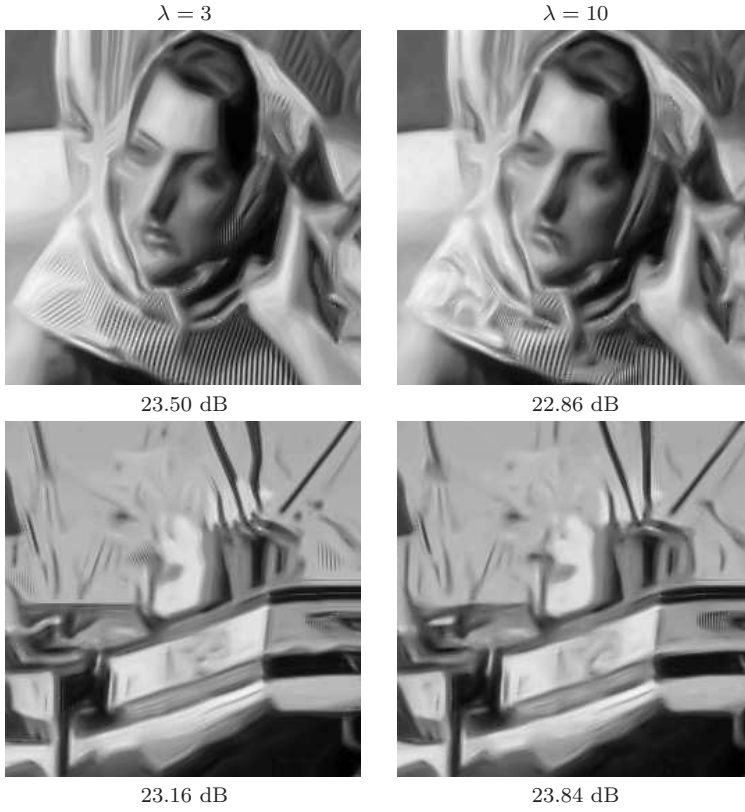


Fig. 4 Results of subsample image restoration from 5% subsamples.

Following the algorithm of LDMM in [15], we need to solve following optimization problem:

$$\min_{\substack{\mathbf{u} \in \mathbb{R}^{m \times n}, \\ \alpha_1, \dots, \alpha_d}} \sum_{i=1}^d \|\nabla_{\mathcal{M}} \alpha_i\|_{L^2(\mathcal{M})}^2 + \gamma \sum_{x \in \bar{\Omega}} |\mathbf{b}(x) - \mathbf{u}(x)|^2, \quad (6.2)$$

subject to:  $\alpha_i(\bar{\mathcal{P}}(\mathbf{u}^k)(x)) = (\mathcal{P}_i \mathbf{u})(x), \quad x \in \bar{\Omega}, \quad i = 1, \dots, d.$



**Fig. 5** Results of subsample image restoration with different weight in the semi-local patches.

where  $\gamma$  is a penalty weight related to the variance of the noise.

Equivalently, we can impose the penalty on the patches,

$$\min_{\substack{\mathbf{u} \in \mathbb{R}^{m \times n}, \\ \alpha_1, \dots, \alpha_d}} \sum_{i=1}^d \|\nabla_{\mathcal{M}} \alpha_i\|_{L^2(\mathcal{M})}^2 + \gamma \sum_{i=1}^d \sum_{x \in \bar{\Omega}} |\alpha_i(\bar{\mathcal{P}}(\mathbf{u}^k)(x)) - (\mathcal{P}_i \mathbf{b})(x)|^2, \quad (6.3)$$

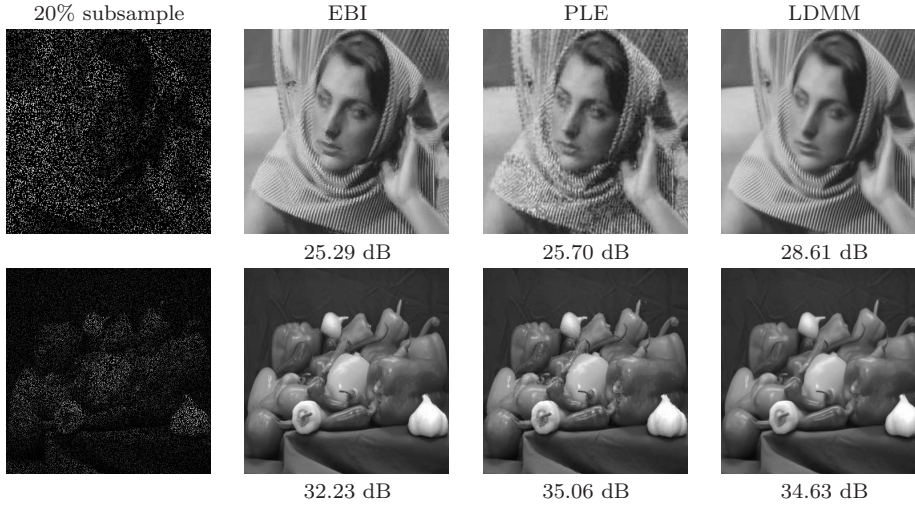
subject to:  $\alpha_i(\bar{\mathcal{P}}(\mathbf{u}^k)(x)) = (\mathcal{P}_i \mathbf{u})(x)$ ,  $x \in \bar{\Omega}$ ,  $i = 1, \dots, d$ .

Next, we split  $\mathbf{u}$  and  $\alpha_1, \dots, \alpha_d$ , and solve them separately.

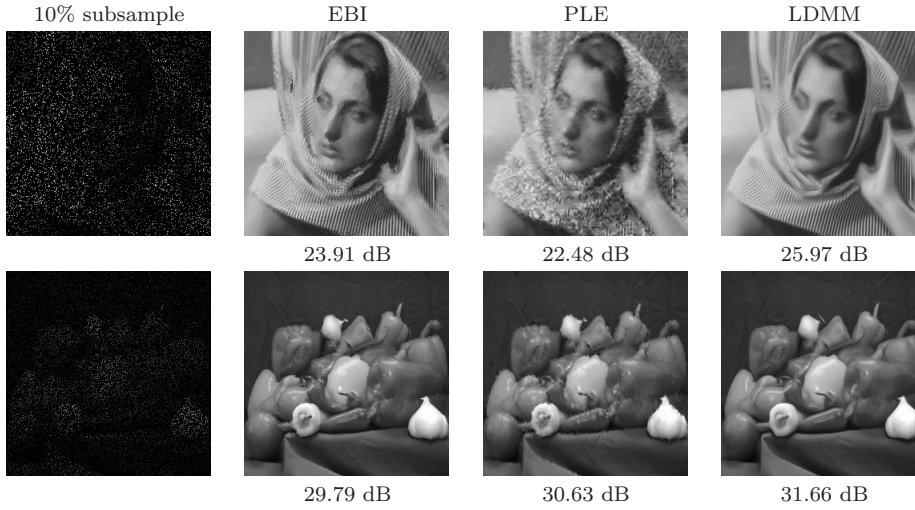
– Compute  $\alpha_i^{k+1}$ ,  $i = 1, \dots, d$  by solving

$$\min_{\alpha_1, \dots, \alpha_d} \sum_{i=1}^d \|\nabla_{\mathcal{M}} \alpha_i\|_{L^2(\mathcal{M})}^2 + \gamma \sum_{i=1}^d \sum_{x \in \bar{\Omega}} |\alpha_i(\bar{\mathcal{P}}(\mathbf{u}^k)(x)) - (\mathcal{P}_i \mathbf{b})(x)|^2 \quad (6.4)$$





**Fig. 6** Results of subsample image restoration from 20% subsamples.



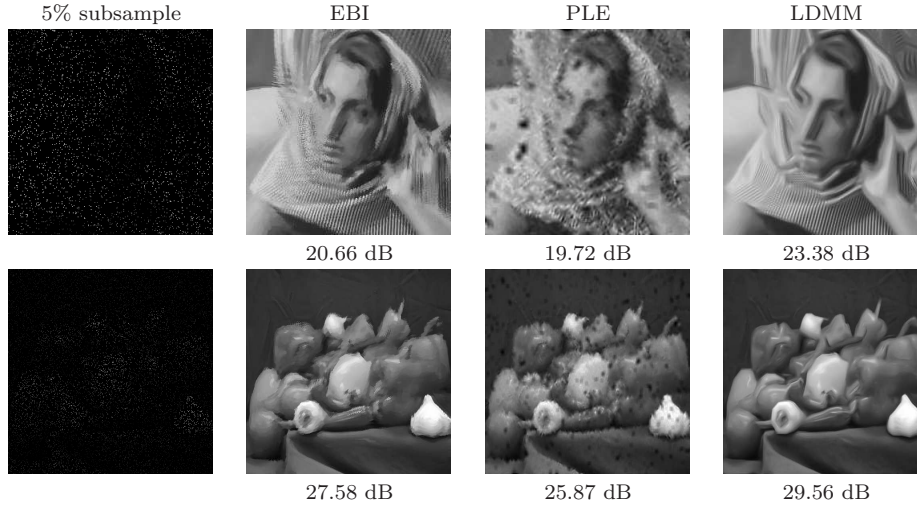
**Fig. 7** Results of subsample image restoration from 10% subsamples.

- Update the image  $\mathbf{u}^{k+1}$  by solving the least-squares problem

$$(\mathcal{P}_i \mathbf{u})(x) = \alpha_i (\bar{\mathcal{P}}(\mathbf{u}^k)(x)), \quad i = 1, \dots, d, \quad x \in \bar{\Omega}. \quad (6.5)$$

In (6.4),  $\alpha_1, \dots, \alpha_d$  are decoupled, we can solve them individually. For each  $\alpha_i$ , we need to solve following type of problem:

$$\min_{u \in H^1(\mathcal{M})} \|\nabla_{\mathcal{M}} u\|_{L^2(\mathcal{M})}^2 + \gamma \sum_{\mathbf{y} \in P} |u(\mathbf{y}) - v(\mathbf{y})|^2, \quad (6.6)$$



**Fig. 8** Results of subsample image restoration from 5% subsamples.

where  $u$  can be any  $\alpha_i$ ,  $\bar{P} = \bar{P}(\mathbf{u}^k)$  is the semi-local patch set of  $\mathbf{u}^k$  and  $v(\mathbf{y})$  is a given function on  $\bar{P}(\mathbf{u}^k)$  corresponding to  $(\mathcal{P}_i \mathbf{b})(x)$ .

By a standard variational approach, the solution of (6.6) can be obtained by solving the following PDE

$$\begin{cases} -\Delta_{\mathcal{M}} u(\mathbf{x}) + \gamma \sum_{\mathbf{y} \in \bar{P}} \delta(\mathbf{x} - \mathbf{y})(u(\mathbf{y}) - v(\mathbf{y})) = 0, & \mathbf{x} \in \mathcal{M} \\ \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = 0, & \mathbf{x} \in \partial \mathcal{M}. \end{cases} \quad (6.7)$$

where  $\partial \mathcal{M}$  is the boundary of  $\mathcal{M}$  and  $\mathbf{n}$  is the out normal of  $\partial \mathcal{M}$ . If  $\mathcal{M}$  has no boundary,  $\partial \mathcal{M} = \emptyset$ .

Using the point integral method, the above Laplace-Beltrami equation is discretized as follows [15],

$$\sum_{\mathbf{y} \in \bar{P}} w(\mathbf{x}, \mathbf{y})(u(\mathbf{x}) - u(\mathbf{y})) + \bar{\gamma} \sum_{\mathbf{y} \in \bar{P}} w(\mathbf{x}, \mathbf{y})(u(\mathbf{y}) - v(\mathbf{y})) = 0, \quad \mathbf{x} \in \bar{P}.$$

$\bar{\gamma}$  is a parameter related to  $\gamma$ . We only need to set  $\bar{\gamma}$  in the computation.

Based on the discussion in Section 3, the equation above is modified to get the equation in WNLL approach,

$$\sum_{\mathbf{y} \in \bar{P}} w(\mathbf{x}, \mathbf{y})(u(\mathbf{x}) - u(\mathbf{y})) + \bar{\gamma} \sum_{\mathbf{y} \in \bar{P}} w(\mathbf{x}, \mathbf{y})(u(\mathbf{x}) - v(\mathbf{y})) = 0, \quad \mathbf{x} \in \bar{P}, \quad (6.8)$$

Notice that (6.8) can be also derived from following variational problem,

$$\min_u \sum_{\mathbf{x}, \mathbf{y} \in \bar{P}} w(\mathbf{x}, \mathbf{y})(u(\mathbf{x}) - u(\mathbf{y}))^2 + \bar{\gamma} \sum_{\mathbf{x}, \mathbf{y} \in \bar{P}} w(\mathbf{x}, \mathbf{y})(u(\mathbf{x}) - v(\mathbf{y}))^2, \quad (6.9)$$

**Algorithm 2** Image Denoising**Require:** A noisy image  $\mathbf{b}$ .**Ensure:** A denoised image  $\mathbf{u}$ .Let  $\mathbf{u}^0 = \mathbf{b}$ .**while** not converge **do**

1. Generate semi-local patch set  $\bar{P}(\mathbf{u}^k)$  from current image  $\mathbf{u}^k$  with the weight  $\lambda$  in the semi-local patch.
2. Compute the weight function  $w(\mathbf{x}, \mathbf{y})$ ,  $\mathbf{x}, \mathbf{y} \in \bar{P}(\mathbf{u}^k)$ .
3. Compute  $\alpha_i$ ,  $i = 1, \dots, d$  by solving (6.8) with  $P = \bar{P}(\mathbf{u}^k)$ ,  $v = (\mathcal{P}_i \mathbf{b})(x)$ .
3. Update the image  $\mathbf{u}^{k+1}$  by

$$\mathbf{u}^{k+1} = \left( \sum_{i=1}^d \mathcal{P}_i^* \mathcal{P}_i \right)^{-1} \left( \sum_{i=1}^d \mathcal{P}_i^* \alpha_i^{k+1} \right)$$

where  $\mathcal{P}_i^*$  is the adjoint operator of  $\mathcal{P}_i$ .

4.  $k \leftarrow k + 1$ .

**end while** $\mathbf{u} = \mathbf{u}^k$ .

Based on the above derivation, we get an iterative algorithm (Algorithm 2) for image denoising.

In the rest of this section, we will show some results on image denoising. We use the same weight given in (5.5). The patch size is  $10 \times 10$ . The number of iterations is 3. The weight  $\lambda$  in the semi-local patches is 5. In our tests, the noise  $\varepsilon$  is Gaussian noise with the standard deviation  $\sigma = 100$ .  $\bar{\gamma} = 0.5$  in (6.8).

The results are shown in Fig. 10. BM3D is the famous method of block-matching with 3D collaborative filtering [8], which is known to give state-of-the-art results in image denoising. As we can see in Fig. 10, the results of BM3D are better in PSNR except for the image of Barbara. LDMM seems to reconstruct the edges and textures sharper than BM3D as shown in Fig. 9. This difference can also be found in the result of the image of the house. LDMM seems to recover the edges of the house smoother and sharper, although the PSNR is lower than that of BM3D.

Actually, the philosophy in LDMM and BM3D is very similar. In Algorithm 2, there are also grouping, filtering and aggregation step as those in BM3D. The main difference is that in LDMM, we use a Laplace-Beltrami operator as a filter directly on the patches while hard thresholding and Wiener filtering is using in BM3D in the transform domain. LDMM provides a general framework for patch-based nonlocal methods. We can also use other operators, such as nonlocal total variation, to replace the Laplace-Beltrami operator in LDMM. In many problems of image processing, nonlocal TV, which corresponds to an  $L_1$  regularization, has been proven to be very effective. In our subsequent research, we are planing to plug nonlocal TV into LDMM. The new model should have even better performance.



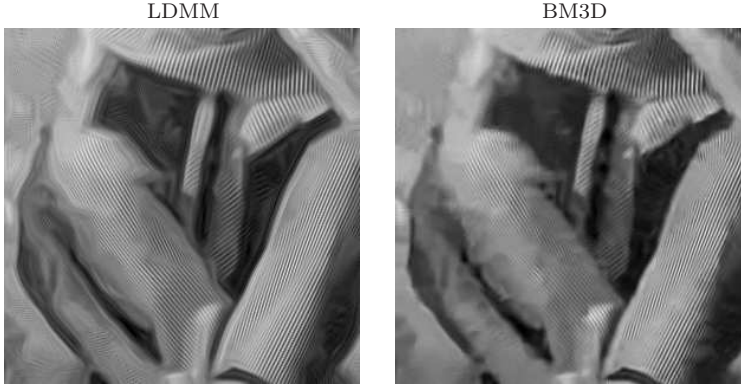


Fig. 9 Fragment of the denoised image of Barbara using LDMM and BM3D.

## 7 Discussion of WNLL with General Constraints

In this paper, we improve the LDMM using the semi-local patches and the weighted nonlocal Laplacian. The new method is shown to be very powerful in image inpainting and denoising. One natural question is if we can apply the idea of WNLL to solve the problem with general constraints. More specifically, we need to solve following optimization problem:

$$\min_u \|\nabla_{\mathcal{M}} u\|_2^2 + \gamma \|\Phi u - b\|_2^2. \quad (7.1)$$

Here  $\Phi$  is a given linear operator.

Following standard variational approach, the optimal solution is given by solving an elliptic equation

$$\begin{cases} -\Delta_{\mathcal{M}} u(x) + \gamma \sum_{\mathbf{y} \in P} \delta(\mathbf{x} - \mathbf{y}) (\Phi^* \Phi u(\mathbf{y}) - \Phi^* b(\mathbf{y})) = 0, & \mathbf{x} \in \mathcal{M} \\ \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = 0, & \mathbf{x} \in \partial \mathcal{M}. \end{cases}$$

where  $\Phi^*$  is the adjoint operator of  $\Phi$ .

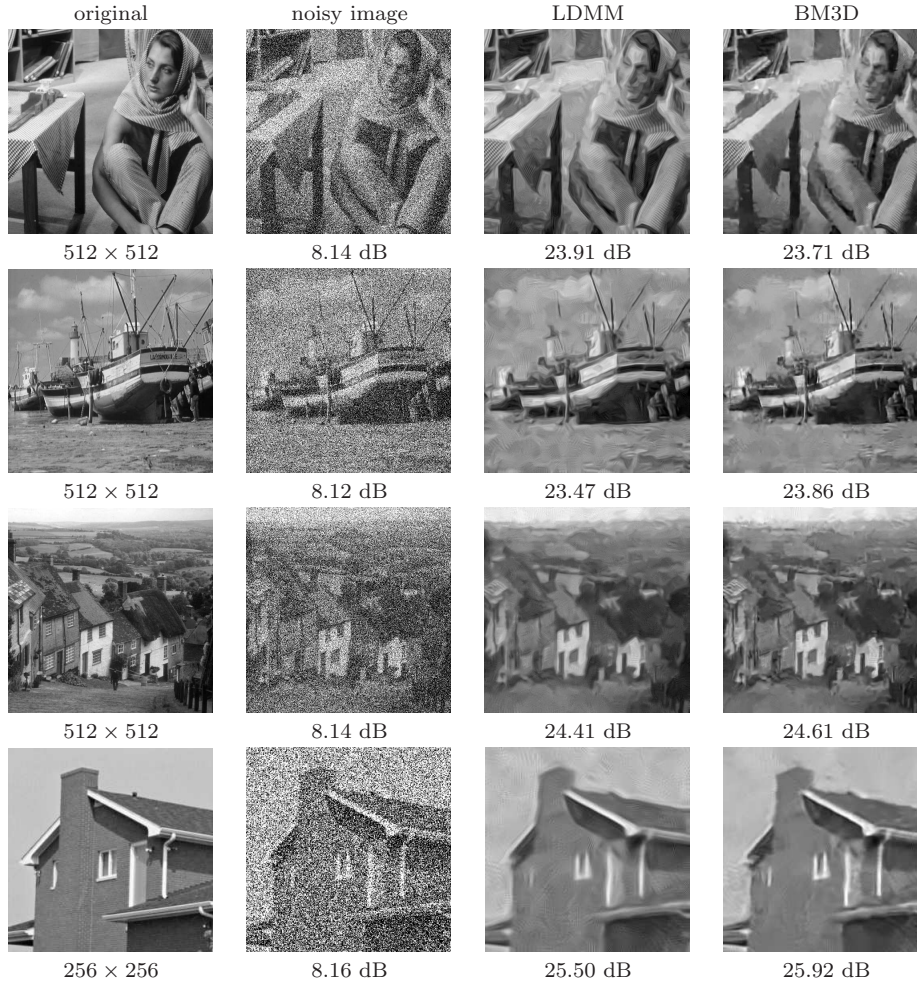
Using the point integral method, above equation is discretized on the point cloud  $P$ ,

$$\sum_{\mathbf{y} \in P} w(\mathbf{x}, \mathbf{y}) (u(\mathbf{x}) - u(\mathbf{y})) + \bar{\gamma} \sum_{\mathbf{y} \in P} w(\mathbf{x}, \mathbf{y}) (\Phi^* \Phi u(\mathbf{y}) - \Phi^* b(\mathbf{y})) = 0, \quad \mathbf{x} \in P.$$

Based on the formal discussion in Section 3, the corresponding WNLL formulation seems to be

$$\sum_{\mathbf{y} \in P} w(\mathbf{x}, \mathbf{y}) (u(\mathbf{x}) - u(\mathbf{y})) + \bar{\gamma} \sum_{\mathbf{y} \in P} w(\mathbf{x}, \mathbf{y}) (\Phi^* \Phi u(\mathbf{x}) - \Phi^* b(\mathbf{y})) = 0, \quad \mathbf{x} \in P.$$

However, this formulation is not right. It is easy to check that it is not consistent to WNLL in the interpolation problem. The reason is very simple.



**Fig. 10** Results of image denoising.

The function,  $\Phi^* \Phi u$ , may not be smooth on  $\mathcal{M}$ . Hence, we can not replace  $\Phi^* \Phi u(\mathbf{y})$  by  $\Phi^* \Phi u(\mathbf{x})$  even if  $\mathbf{x}$  is close to  $\mathbf{y}$ .

One possible WNLL formulation with general constraints may be given by split Bregman. First, we introduce an auxiliary variable  $v$  and rewrite (7.1) in an equivalent form

$$\min_{u,v} \|\nabla_{\mathcal{M}} u\|_2^2 + \gamma \|\Phi v - b\|_2^2, \quad \text{subject to: } u = v \quad (7.2)$$

The above optimization problem is solved by split Bregman iteration.

- $u^{k+1} = \arg \min_u \|\nabla_{\mathcal{M}} u\|_2^2 + \mu \|u - v^k + d^k\|_2^2$
- $v^{k+1} = \arg \min_v \mu \|u^{k+1} - v + d^k\|_2^2 + \gamma \|\Phi v - b\|_2^2$

$$-d^{k+1} = d^k + u^{k+1} - v^{k+1}.$$

$v$  and  $d$  are solved by standard least-squares and simple algebraic calculation. The key step in the above iteration is to solve  $u$ . Here,  $u$  is solved by the WNLL approach,

$$\sum_{\mathbf{y} \in P} w(\mathbf{x}, \mathbf{y})(u(\mathbf{x}) - u(\mathbf{y})) + \bar{\mu} \sum_{\mathbf{y} \in P} w(\mathbf{x}, \mathbf{y})(u(\mathbf{x}) - v^k(\mathbf{y}) + d^k(\mathbf{y})) = 0, \quad \mathbf{x} \in P$$

This seems to be a possible way to apply the idea of WNLL to general image processing problem. On the other hand, we are also working to integrate different methods, such as nonlocal TV,  $\infty$ -Laplacian, within the framework of LDMM.

## References

1. M. Aharon, M. Elad, and A. Bruckstein. K-svd: An algorithm for designing overcomplete dictionaries for sparse representation. *IEEE Trans. Signal Process.*, 54:4311–4322, 2006.
2. P. Arias, V. Caselles, and G. Sapiro. A variational framework for non-local image inpainting. *Proceedings of 7th Int. Conf. on EMMCPVR*, 2009.
3. A. Buades, B. Coll, and J.-M. Morel. A non-local algorithm for image denoising. *IEEE Computer Society Conference on Computer Vision and Pattern Recognition(CVPR)*, 2:60–65, 2005.
4. A. Buades, B. Coll, and J.-M. Morel. A review of image denoising algorithms, with a new one. *Multiscale Model. Simul.*, 4:490–530, 2005.
5. G. Carlsson, T. Ishkhanov, V. de Silva, and A. Zomorodian. On the local behavior of spaces of natural images. *International Journal of Computer Vision*, 76:1–12, 2008.
6. P. Chatterjee and P. Milanfar. Patch-based near-optimal image denoising. *Image Processing, IEEE Transactions on*, 21:1635–1649, 2012.
7. D. Comaniciu and P. Meer. Mean shift: A robust approach toward feature space analysis. *IEEE Trans. Pattern Analysis and Machine Intelligence*, 24:603–619, 2002.
8. K. Dabov, A. Foi, V. Katkovnik, and K. Egiazarian. Image denoising by sparse 3d transform-domain collaborative filtering. *IEEE Trans. Image Process.*, 16:2080–2095, 2007.
9. V. C. G. Facciolo, P. Arias and G. Sapiro. Exemplar-based interpolation of sparsely sampled image. *Proceedings of 7th Int. Conf. on EMMCPVR*.
10. G. Gilboa and S. Osher. Nonlocal operators with applications to image processing. *Multiscale Model. Simul.*, 7:1005–1028, 2008.
11. A. Kheradmand and P. Milanfar. A general framework for regularized, similarity-based image restoration. *Image Processing, IEEE Transactions on*, 23:5136–5151, 2014.
12. A. B. Lee, K. S. Pedersen, and D. Mumford. The nonlinear statistics of high-contrast patches in natural images. *International Journal of Computer Vision*, 54:83–103, 2003.
13. Z. Li and Z. Shi. A convergent point integral method for isotropic elliptic equations on point cloud. *SIAM: Multiscale Modeling Simulation*, 14:874–905, 2016.
14. Z. Li, Z. Shi, and J. Sun. Point integral method for solving poisson-type equations on manifolds from point clouds with convergence guarantees. *Communications in Computational Physics*, 22:228–258, 2017.
15. S. Osher, Z. Shi, and W. Zhu. Low dimensional manifold model for image processing. *to appear in SIAM Journal on Imaging Sciences*, 2017.
16. J. A. Perea and G. Carlsson. A klein-bottle-based dictionary for texture representation. *International Journal of Computer Vision*, 107:75–97, 2014.
17. G. Peyré. Image processing with non-local spectral bases. *Multiscale Model. Simul.*, 7:703–730, 2008.
18. G. Peyré. Manifold models for signals and images. *Computer Vision and Image Understanding*, 113:248–260, 2009.

19. G. Peyré. A review of adaptive image representations. *IEEE Journal of Selected Topics in Signal Processing*, 5:896–911, 2011.
20. Z. Shi, S. Osher, and W. Zhu. Weighted nonlocal laplacian on interpolation from sparse data. *to appear in Journal of Scientific Computing*, 2017.
21. Z. Shi and J. Sun. Convergence of the point integral method for poisson equation on point cloud. *to appear in Research in the Mathematical Sciences*, 2017.
22. Z. Shi, J. Sun, and M. Tian. Harmonic extension on point cloud. *mathscidoc:1609.19003*.
23. S. M. Smith and J. M. Brady. Susan - a new approach to low level image processing. *International Journal of Computer Vision*, 23:45–78, 1997.
24. A. Spira, R. Kimmel, and N. Sochen. A short time beltrami kernel for smoothing images and manifolds. *IEEE Trans. Image Processing*, 16:1628–1636, 2007.
25. H. Talebi and P. Milanfar. Global image denoising. *Image Processing, IEEE Transactions on*, 23:755–768, 2014.
26. C. Tomasi and R. Manduchi. Bilateral filtering for gray and color images. *Proceedings of the Sixth International Conference on Computer Vision*, page 839, 1998.
27. Y. Wang, A. Szlam, and G. Lerman. Robust locally linear analysis with applications to image denoising and blind inpainting. *SIAM J. Imaging Sciences*, 6:526–562, 2013.
28. R. Yin, T. Gao, Y. M. Lu, and I. Daubechies. A tale of two bases: Local-nonlocal regularization on image patches with convolution framelets. *arXiv:1606.01377*, 2016.
29. G. Yu, G. Sapiro, and S. Mallat. Solving inverse problems with piecewise linear estimators: From gaussian mixture models to structured sparsity. *IEEE Transactions on Image Processing*, 21.
30. G. Yu, G. Sapiro, and S. Mallat. Image modeling and enhancement via structured sparse model selection. *Proceedings of the 17th IEEE International Conference on Image Processing (ICIP)*, 2010.
31. M. Zhou, H. Chen, J. Paisley, L. Ren, L. Li, Z. Xing, D. Dunson, G. Sapiro, and L. Carin. Nonparametric bayesian dictionary learning for analysis of noisy and incomplete images. *IEEE Trans. Image Processing*, 21:130–144, 2012.
32. D. Zoran and Y. Weiss. From learning models of natural image patches to whole image restoration. In *Int. Conf. on Computer Vision*, 2011.